Mathematical Reviews

UNIVERSITY OF MICHIGAN
APR 7 1959
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Vol. 19, No. 11 December, 1958 pp. 1151-1248

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Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) 4(82) (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place.

Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, 190 Hope St., Providence 6, R.I.

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Editorial Office

MATHEMATICAL REVIEWS, 190 Hope St., Providence 6, R.I.

Subscription: Price $50 per year ($25 per year to individual members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to the American Mathematical Society, 190 Hope St., Providence 6, R.I.

The preparation of the reviews appearing in this publication is made possible by support provided by a grant from the National Science Foundation and by a contract with the Air Force Office of Scientific Research, Air Research and Development Command, United States Air Force. The publication was initiated with funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers or proprietors of the publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Published for the American Mathematical Society by N.V. Erven P. Noordhoff, Uitgeversmaak, Groningen, the Netherlands.

Mathematical Reviews

Vol. 19, No. 11 DECEMBER, 1958 pp. 1151-1248

FOUNDATIONS, THEORY OF SETS, LOGIC

Levi, Beppo. Some reflections on mathematics and philosophy. Math. Notae **14** (1955), 133–140 (1956). (Spanish)

These are the sketchy thoughts of an Italian mathematician, long resident in the Argentine and, since 1941, editor of "Mathematicae Notae". For Levi, science and mathematics are concerned with "what is external to ourselves," philosophy with "what we feel within ourselves" (p. 133). But he makes no effort to sharpen this rather vague distinction, nor to explore the relation between mathematics and philosophy, thus understood.

Levi dissents from the usual view that mathematics and the empirical sciences can look forward to many centuries of further development. Rather, he suggests, "the high point [of scientific development] may already have been passed" (p. 139). Mathematicians are dissipating their energies upon such minor "embroideries" as mathematical logic and set theory (Levi's own field is mathematical analysis). Physicists and chemists have been startled by problems whose very existence they long failed to suspect; and many such problems are still without intellectually satisfying answers. Levi suggests (without explicitly stating) that such problems may become more rather than less troublesome as time goes by.

G. L. Kline (New York, N.Y.).

Dingler, Hugo. La ricostruzione dell'aritmetica. I, II, III. Methodos **7** (1955), 247–287; **8** (1956), 95–137, 177–199.

Enrico Albani and the editors of "Methodos" have performed a genuine service in presenting this Italian translation, with introduction and notes, of Ch. III of the unpublished "Lehrbuch der exakten Fundamentalwissenschaft" of Hugo Dingler (1881–1954). (Chs. I and II are also summarized briefly in the translator's introduction.) The complete German text of Ch. III is appended, but, unfortunately, is marred by an excessive number of misprints, some of which are seriously misleading.

Dingler's "reconstruction of arithmetic" starts with unity, defined as a "discriminable constancy," and from it builds up what he calls "Einsen-Zahlen" (*E*-Zahlen), as distinguished from "Zähl-Zahlen" (*Z*-Zahlen). Such "unity-numbers", plus the operations of addition and multiplication, generate a "Regeln-Kalkül" (*R*-Kalkül), based on the ten rules governing these operations (II, 128). Among them are the ordinary commutative laws: $a+b=b+a$ and $a\cdot b=b\cdot a$ (Rules 2 and 4). Subtraction is simply addition using negative terms (*n*-Terme), written a' (or $-a$); division is multiplication using "division-terms" (*d*-Terme), written a'' (or a^{-1}). Thus: $a+(b+b')=a$; $a\cdot(b\cdot b'')=a$ (Rules 7 and 9).

The next step is to restrict the *R*-calculus in such a way that $a=b$ is not deducible (by prohibiting the use of expressions like $(c+c')''$). Dingler calls the restricted calculus a *W*-Kalkül or "grössenfreie Arithmetik" (since it does not use the relations $>$ or $<$), and claims that it is demonstrably consistent. {Lorenzen pointed out a flaw in this alleged consistency-proof [Gedenkbuch für Hugo Dingler, Münich, 1956, pp. 127–128].} Dingler sees the advantage of his own procedure in the systematic derivation, rather than mere postulation, of unity (the Eins), and in the further derivation of the rules of arithmetical operation (III, 191).

Detailed technical criticism of Dingler's undertaking should, in the reviewer's opinion, be deferred until his book has appeared, at which time his "reconstruction" of arithmetic can be seen in its relation to the "reconstruction" of geometry, kinematics, and mechanics. One or two brief comments on the translation may be in order: Albani occasionally blurs Dingler's distinctions, e.g., rendering both "auftreten" and "dasein" by "presentarsi" (I, 254, 255) and both "Dasein" and "Vorhandensein" by "esistenza" (I, 255, 267). And he takes undue liberties in rendering "Willensvorgang" by "scelta" (I, 266). To say: "Il *W*-calcolo, come il calcolo aritmetico, è esente da contraddizzione," is to misrepresent Dingler's "Der *W*-Kalkül ist als arithmetischer Kalkül widerspruchsfrei" (II, 120, 136). Generally speaking, however, the more technical passages are rendered with admirable clarity and fidelity to the original. Albani's strictures (concentrated in part I) are sometimes helpful, although they are based on a narrowly "operational" approach to the philosophy of mathematics.

G. L. Kline (New York, N.Y.).

Schröter, Karl. Eine Umformung des Heytingschen Axiomensystems für den intuitionistischen Aussagenkalkül. Z. Math. Logik Grundlagen Math. **3** (1957), 18–29.

Complete proof of the equivalence (first proved by the author in 1937, but unpublished) of a system *S* of axioms for the intuitionistic propositional calculus *IP* with the system *H*, given by the reviewer [S.-B. Preuss. Akad. Wiss. **1930**, 42–56]. *S* consists of four groups of axioms; I contains only $\rightarrow$, II $\rightarrow$ and $\wedge$, III $\rightarrow$ and $\vee$, IV $\rightarrow$ and $\neg$. From I follow all the theorems of *IP* in $\rightarrow$ alone, from II those in $\rightarrow$ and $\wedge$, from III those in $\rightarrow$ and $\vee$, from IV those in $\rightarrow$ and $\neg$ [Wajsberg, Wiadom. Mat. **46** (1939), 45–101]. It follows from the proof that I and II together are equivalent with axioms 1–6 from *H*, with analogous equivalences for the axioms in $\rightarrow$ and $\vee$, and in $\rightarrow$ and $\neg$, so that completeness properties for *H* follow.

A. Heyting (Amsterdam).

Weston, J. D. A short proof of Zorn's lemma. Arch. Math. **8** (1957), 279.

This reviewer found the proof somewhat too short.

R. M. Baer (Berkeley, Calif.).

Erdös, P.; and Fodor, G. Some remarks on set theory. VI. Acta Sci. Math. Szeged **18** (1957), 243–260.

Let E be a given uncountable set of power m and let R be a relation on E. For x in E, let $R(x)$ denote the set of elements y in E for which xRy holds. Two distinct elements of E, x and y, are called independent if $x \notin R(y)$ and $y \notin R(x)$. A subset F of E is called free if F has only one element, or if F has more than one element and each two are independent. Let B be a system of subsets of E and I a $\mathfrak{p}$-additive ideal of B, $\mathfrak{p} \leq m$. (A non-empty subset $I \subset B$ is a $\mathfrak{p}$-additive ideal if the sum of any system of power smaller than $\mathfrak{p}$, of elements of I, is also in I, and if $X \in I$, $Y \in B$, $Y \subset X$ imply $Y \in I$.) Let $\{x\} \in B$ and $\{x\} \in I$ for every $x \in E$. Let one of the following conditions hold for the sets $R(x)$: (A) There is a cardinal number $n < m$ such that $|R(x)| < n$ for every x in E; (B) E is a metric space and $d(x, R(x)) > 0$, where $d(x, R(x))$ is the distance from x to the set $R(x)$.

Numerous results about the following problem are given. (i) If A is a system of sets of $B - I$, does there exist a free subset E' of E such that $X \cap E' \in B - I$ for every $X \in A$? For example, an affirmative answer is given in the case where $m > \aleph_0$ is less than the first weakly inaccessible aleph, $B = 2^E$, I is an $\aleph_{\gamma+1}$-additive ideal ($\aleph_{\gamma+1} \leq m$), $|A| = \aleph_0$, and $|R(x)| < \aleph_0$ for every x in E. A second problem is also considered: (ii) Let K be a class of subsets of E. When does there exist a relation R for which condition (A) holds and there is no free subset $X \in K$ with respect to R? A sample result obtained pertaining to (ii) is the following: If $|K| = m$ and every element of K is of power m, then there exists a relation R, with $|R(x)| \leq 1$ for every x in E, for which there is no free set in K. *S. Ginsburg* (Hawthorne, Calif.).

de Bruijn, N. G. A theorem on choice functions. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 409–411.

The theorem alluded to in the title is the following. Let M be an infinite (index) set of power m. Let $\{A_u; u \in M\}$ be a family of non-empty sets with the property that for each positive integer n there are m indices u such that $|A_u| > n$. Then there exists a set F^*, of power 2^m, of choice functions whose members are independent. Four remarks on applications and extensions of the theorem are then given. {It appears to the reviewer that remark 3, which pertains to the independence of a certain portion of the proof from the axiom of choice, is incorrect since, if m is an arbitrary cardinal number, $m^2 = m$ depends on the axiom of choice.}

Notation: f is a choice function if f is defined on M such that $f(u)$ is in A_u for each u. The members of a set F^* of choice functions are said to be independent if, for each finite number of elements $f_1, \cdots, f_k$ of F^*, there is an index u such that $f_1(u), \cdots, f_k(u)$ are pairwise different. *S. Ginsburg* (Hawthorne, Calif.).

Bruns, Günter. Zur Struktur von Filtern. Math. Ann. **134** (1958), 205–224.

For a filter $\mathfrak{F}$ on a set E, the four cardinal numbers $t(\mathfrak{F})$, $df(\mathfrak{F})$, $z(\mathfrak{F})$ and $s(\mathfrak{F})$ are defined as follows. $t(\mathfrak{F})$ is the least cardinal number $|F|$ for $F \in \mathfrak{F}$. $df(\mathfrak{F}) = \min_{F \in \mathfrak{F}} df(F, \mathfrak{F})$, where $df(F, \mathfrak{F})$ is the immediate successor of all cardinal numbers $|F - F'|$ with $F' \in \mathfrak{F}$. $z(\mathfrak{F}) = \min_{F \in \mathfrak{F}} z(F, \mathfrak{F})$; here $z(F, \mathfrak{F})$ denotes the least cardinal number m for which there exists no partition $F = \bigcup A_\nu$ of F such that $|A_\nu| \geq m$ and $E - A_\nu \in \mathfrak{F}$ for each ν. $s(\mathfrak{F})$ is defined only for non-principal filters $\mathfrak{F}$: it is the least cardinal number m for which there exists a subsystem $\mathfrak{G} \subset \mathfrak{F}$ with $|\mathfrak{G}| = m$ and $\bigcap_{G \in \mathfrak{G}} G \notin \mathfrak{F}$. For four cardinal numbers $\aleph_\alpha$, $\aleph_\beta$, $\aleph_\gamma$, $\aleph_\delta$, the author gives a necessary and sufficient set of conditions in order that there exist a filter $\mathfrak{F}$ with empty intersection and with the preassigned $t(\mathfrak{F}) = \aleph_\alpha$, $df(\mathfrak{F}) = \aleph_\beta$, $z(\mathfrak{F}) = \aleph_\gamma$, $s(\mathfrak{F}) = \aleph_\delta$. Another main result is concerned with a representation of a filter $\mathfrak{F}$ as the direct intersection of two filters $\mathfrak{G}$, $\mathfrak{H}$; i.e., $\mathfrak{F} = \mathfrak{G} \cap \mathfrak{H}$ such that some set of $\mathfrak{G}$ is disjoint from some set of $\mathfrak{H}$. Here it is understood that the power set $\mathfrak{P}(E)$ of E is considered as a filter on E. Let $\mathfrak{C}_\alpha = \{F \subset E : |E - F| < \aleph_\alpha\}$. For $M \subset E$, let $\mathfrak{C}_\alpha(M)$ denote the filter on E generated by $\{A \cap M : A \in \mathfrak{C}_\alpha\}$. Then the theorem states that any filter $\mathfrak{F}$ on E finer than $\mathfrak{C}_\alpha$ can be expressed as a direct intersection $\mathfrak{F} = \mathfrak{C}_\alpha(M) \cap \mathfrak{G}$, where M is a suitable subset of E and $\mathfrak{G}$ is a filter on E with $z(\mathfrak{G}) \geq \aleph_{\alpha+1}$. (In case $\mathfrak{G} = \mathfrak{P}(E)$, the relation $z(\mathfrak{G}) \geq \aleph_{\alpha+1}$ is regarded as fulfilled.) This paper is a sequel to the author's earlier one [Math. Ann. **133** (1957), 26–38; MR **19**, 49].

Ky Fan (Notre Dame, Ind.).

Wagner, K. Verbandstheoretische Charakterisierung der Cantorschen Äquivalenzrelation. Math. Ann. **134** (1958), 295–297.

Let S be a family of sets. Suppose that S contains all subsets of each of its elements. It is shown that the Cantor equivalence relation "$\sim$", i.e., $A \sim B$ if A and B have the same power, is the finest equivalence relation on S among all equivalence relations on S which satisfy two technical conditions. *S. Ginsburg* (Hawthorne, Calif.).

Steiner, H. G. Einführung in die Relationentheorie. Math.-Phys. Semesterber. **5** (1957), 261–271.

This is an introduction to the basic concepts of the (intuitive) theory of relations, including a discussion of congruence relations and the derived quotient structures.

E. Mendelson (New York, N.Y.).

Ono, Katuzi. On some properties of binary relations. Nagoya Math. J. **12** (1957), 161–170.

A binary relation R is said to be closed if $RR^{-1}R = R$. This means that R is the union of cartesian products $A_t \times B_t$ where the sets A_t as well as the sets B_t are pairwise disjoint, or equivalently that $R = F^{-1}G$, where F and G are functions. Some elementary properties of this notion are obtained, e.g., if R and S are closed and if $R \subseteq S$, then $RS^{-1}R$ and $SR^{-1}S$ are also closed, and there exist functions F, G, F_0, G_0, F_1, G_1, H such that $R = F_0^{-1}G_0$, $RS^{-1}R = F_0^{-1}H^{-1}HG_0$, $SR^{-1}S = F_1^{-1}G_1$, $S = F^{-1}G$, $F_0F_0^{-1} = G_0G_0^{-1}$, $F_1F_1^{-1} = G_1G_1^{-1} = HF_0F_0^{-1}H^{-1} = HG_0G_0^{-1}H^{-1}$, $FF^{-1} = GG^{-1}$, $F \supseteq F_1 \supseteq HF_0$, $G \supseteq G_1 \supseteq HG_0$, $FF_1^{-1}F = F_1$, $GG_1^{-1}G = G_1$. *B. Jónsson* (Minneapolis, Minn.).

Gautam, N. D. The validity of equations of complex algebras. Arch. Math. Logik Grundlagenforsch. **3** (1957), 117–124.

It is shown that in order for a (non-trivial) equation to have the property that whenever it is valid in a given algebra $\mathfrak{A}$ it is also valid in the complex algebra of $\mathfrak{A}$, it is necessary and sufficient that each variable which occurs in the equation occur exactly once on each side of the equation. *B. Jónsson* (Minneapolis, Minn.).

See also: Groups and Generalizations: Novikov. **Computing Machines:** Culbertson.

ALGEBRA

Lowig, H. F. J. On the existence of freely generated algebras. Proc. Cambridge Philos. Soc. **53** (1957), 790–795.

This paper continues the painstakingly exact work on the definition of abstract algebras which the author started in J. Reine Angew. Math. **190** (1952), 65–74 [MR 14, 443]. It uses the same comprehensive notation as that paper. Its end is to overcome certain logical difficulties in the transfinite constructions necessary in proving the existence of free algebras. *H. A. Thurston* (Bristol).

Combinatorial Analysis

Ryser, H. J. The term rank of a matrix. Canad. J. Math. **10** (1958), 57–65.

Let $\mathfrak{A}$ be the class of $m\times n$ matrices of 0's and 1's with row sums $r_1, \cdots, r_m$ where $r_1\geqq r_2\geqq\cdots\geqq r_m=1$ and column sums $s_1, \cdots, s_n$ where $s_1\geqq s_2\geqq\cdots\geqq s_n\geqq 1$ and $r_1+\cdots+r_m=s_1+\cdots+s_n=N$. The term rank ρ of a matrix A is the order of the greatest minor of A with a non-zero term in its determinant expansion. This paper finds an expression for the maximal term rank $\bar{\rho}$ of any matrix in the class $\mathfrak{A}$. Let $\bar{s}_i'(i=1, \cdots, n)$ be the number of r's at least $i+1$ (i.e. $\bar{s}_1', \cdots, \bar{s}_n'$ form the partition of $N-m$ dual to $(r_1-1)+\cdots+(r_m-1)=N-m$). Put $s_i'=s_i-1$ $(i=1, \cdots, n)$ and also $s_0'=\bar{s}_0'=0$. Then the expression for $\bar{\rho}$ is $\bar{\rho}=m-M$ where M is the largest integer of the set $\sum_{i=0}^{i}(s_i'-\bar{s}_i')$ $(k=0, 1, \cdots, n)$.

Marshall Hall, Jr. (Columbus, Ohio).

See also: Computing Machines: Culbertson.

Linear Algebra

Albert, A. A. On the orthogonal equivalence of sets of real symmetric matrices. J. Math. Mech. **7** (1958), 219–235.

Let $A_1, \cdots, A_m$ be n-rowed real symmetric matrices and V be the vector space of all matrices $a_1A_1+\cdots+a_mA_m$ for real numbers $a_1, \cdots, a_m$. Let V^* be the vector space generated similarly by a second set of n-rowed real symmetric matrices $A_1^*, \cdots, A_m^*$ and let S be the linear mapping of V onto V^* which maps A_i into A_i^* for $i=1, \cdots, m$. Also suppose that V is Jordan closed (i.e., A, B in V entails $AB+BA$ in V); then

$$\left|xI-\sum_{i=1}^{m}x_iA_i\right|=\left|xI-\sum_{i=1}^{m}x_iA_i^*\right|$$

identically in $x, x_1, \cdots, x_m$, if and only if S is a Jordan algebra isomorphism of V onto V^*. Further, if the conditions of the above theorem hold, the author gives necessary and sufficient conditions for the existence of a real orthogonal matrix P such that $PA_iP'=A_i^*$ for $i=1, \cdots, m$. *H. S. A. Potter* (Aberdeen).

Bodewig, E. Zum Matrizenkalkul. IV. Der Satz von Frobenius. Nederl. Akad. Wetensch. Proc. Ser. A. **60**=Indag. Math. **19** (1957), 82–87.

The theorem of Frobenius referred to states that the matrix $f(A)$ has the same eigenvectors as the matrix A, and if A has eigenvalues λ_i, $i=1, \cdots, n$, then $f(A)$ has eigenvalues $f(\lambda_i)$, provided that the $|\lambda_i|$ are less than the radius of convergence of the power series expansion of the function $f(x)$. If the elementary divisors of A are linear, then any vector u is a linear combination of the eigenvectors x_i: $u=\sum_{i=1}^{n}a_ix_i$; and $f(A)u=\sum_{i=1}^{n}a_if(\lambda_i(x_i$. For a suitable choice of f, and k sufficiently large, any set of components x_i may be effectively eliminated from $[f(A)]^ku$. The author discusses the usefulness of this procedure for finding numerically the eigenvectors and eigenvalues of matrices. Functions considered are: x, $(x-\nu)^{-1}$, ν a constant, e^x, $\log x$, and $\sin x$. *B. N. Moyls.*

Kostant, Bertram. A theorem of Frobenius, a theorem of Amitsur-Levitski and cohomology theory. J. Math. Mech. **7** (1958), 237–264.

The paper is mainly concerned with the following three theorems: (I) The algebra $\mathfrak{g}$ of all $n\times n$ (complex) matrices satisfies the standard identity of degree $2n$, i.e.

$$[A_1\cdots A_{2n}]=\sum \mathrm{sg}(\sigma)A_{\sigma(1)}\cdots A_{\sigma(2n)}=0$$

for any $A_i\in\mathfrak{g}$, where the sum ranges over all permutations σ of $1, \cdots, 2n$, and $\mathrm{sg}(\sigma)=1$ or -1 according as σ is even or odd. (II) Let I be the group algebra (over the complex field) of the symmetric group S_{2n+1} and let α be the cycle $(1, \cdots, 2n+1)$. Then the element $\sum \mathrm{sg}(\rho)\rho\alpha\rho^{-1}$, where ρ runs over S_{2n+1}, belongs to the 2-sided ideal of I generated by the element $(\sum\sigma)(\sum \mathrm{sg}(\tau)\tau)$, where σ runs over all permutations leaving $n+2, \cdots, 2n+1$ invariant, while τ runs over all permutations leaving $2, \cdots, n+1$ invariant. (III) For the Lie algebra $\mathfrak{g}$ (see (I)) and an odd integer r, the unique (up to scalar multiples) invariant cocycle b_r in the primitive class $P^r(g)$ is given by

$$\langle b_r, A_1\wedge\cdots\wedge A_r\rangle=\mathrm{tr}[A_1\cdots A_r],$$

where $A_i\in\mathfrak{g}$. The theorem (I) is due to Amitsur and Levitski [Proc. Amer. Math. Soc. **1** (1950), 449–463; MR 12, 155], (II) is a reformulation by the author of a special case of a theorem of Frobenius on the characters of alternating groups, and (III) is due to Dynkin [Mat. Sb. N.S. **35**(77) (1954), 129–173; MR **16**, 673].

The author first derives (I) from (III) and a theorem of Koszul-Samelson on the behavior of primitive cohomology classes of Lie algebras under homomorphisms. The technique involved proves also that the algebra of $n\times n$ skew-symmetric matrices satisfies the standard identity of degree $2n-2$ when n is even. The author then proves the equivalence of (I) and (II) by an elementary argument. In the last section the author gives, among others, a new proof of (III) which is independent of (I) and (II). The proof is based on the onto isomorphisms:

$$\varphi_r: H_r(\mathfrak{g})\to B(H),\quad \psi^r: B(H)\to H^r(\mathfrak{g}),$$

where H is the subspace of all elements of the group algebra I of S_r of the form $\sum \mathrm{sg}(\rho)\rho f\rho^{-1}$ (the sum runs over all $\rho\in S_r$, and $f\in I$), and where B designates the representation of S_r on the space ${}^rU$ of r-tensors over U (the space on which $\mathfrak{g}$ operates) given by

$$B(\sigma)x_1\otimes\cdots\otimes x_r=x_{\sigma^{-1}(1)}\otimes\cdots\otimes x_{\sigma^{-1}(r)}.$$

The above isomorphisms φ_r and ψ^r are obtained by considering the linear mapping η_r defined by

$$\eta_r(A_1\wedge\cdots\wedge A_r)=\frac{1}{r!}\sum A_{\sigma^{-1}(1)}\otimes\cdots\otimes A_{\sigma^{-1}(r)},$$

where $A_i\in\mathfrak{g}$ and the sum ranges over all $\sigma\in S_r$, and by using a theorem of Weyl to the effect that $B(I)$ is the set of all operators on ${}^rU$ which commute with all $A\otimes\cdots\otimes A$, where A runs over all nonsingular operators on U.

R. Ree (Vancouver, B.C.).

Goldberg, K. Random notes on matrices. J. Res. Nat. Bur. Standards **60** (1958), 321–325.

Four theorems are proved. First, let the elements of a triple-diagonal, symmetric matrix satisfy

$$a_{i,i-1}{}^2=a_{11}-a_{ii}=1-(2s+1)i^{-1}+i^{-2}p(i^{-1}),$$

where $p(t)$ is a polynomial and s a real number not a negative integer. If $D_n(t)$ is the characteristic polynomial of the first principal submatrix of order n, then

$$\lim_{n\to\infty} D_n(a_{11}+1-n^{-2}z^2)=\Lambda_s(z),\ |z|<\infty,$$

where $\Lambda_s(z)$ is the modified Bessel function.

Next, let A be a nonsingular payoff matrix of order n such that the non-null row and column sums of A^{-1} are all of the same sign. Then if none of the row sums is zero, the row player has a unique optimal strategy.

The remaining two theorems are related to each other but not to the first two. One gives a necessary and sufficient condition that two matrices of order n have equal corresponding principal minors. The condition, however, is not easily stated in a few words.

A. Householder (Oak Ridge, Tenn.).

Parodi, Maurice. Localisation des zéros de la dérivée du polynome caractéristique d'une matrice. Bull. Sci. Math. (2) **81** (1957), 119–123.

Let $A=(a_{ij})$ be a square matrix of order n and $f(z)$ its characteristic polynomial; then, if the elements of A are complex, the zeros of $f'(z)$ lie in the domain D' bounded by the least convex curve containing the n circles C_i

$$|z-a_{ii}|\leqq\sum_{j\neq i}|a_{ij}|=P_i\ (i=1,\cdots,n).$$

If the elements of A are real, the zeros of $f'(z)$ lie in the intersection of D' with the domain Ω formed by the union of the elliptic regions E_i

$$\tfrac{1}{4}(x-a_{ii})^2+y^2\leqq P_i^2\ (i=1,\cdots,n),$$

subject to the condition that if one of the circles C_i is disjoint from the others we omit the corresponding ellipse E_i in forming Ω. This result is generalized for the zeros of $g(d/dz)f(z)$, where $g(z)$ is a polynomial with all its zeros real.

H. S. A. Potter (Aberdeen).

Azpeitia, A. G. A generalization of the Vandermonde-Cauchy determinant. Rev. Acad. Ci. Madrid **51** (1957), 149–159 (1 insert). (Spanish)

Let $\rho_1, \rho_2, \cdots, \rho_k$, be arbitrary numbers; let $m_1, m_2, \cdots, m_k$, be integers. Set $m=m_1+m_2+\cdots+m_k$. Define k matrices A_i $(i=1, 2, \cdots, k)$ so that the element $a_{pq}{}^{(i)}$ in row p and column q of A_i is given by $a_{pq}{}^{(i)}=\rho_i{}^{p-1}(p-1)^{q-1}$, where $p=1, 2, \cdots, m$; $q=1, 2, \cdots, m_i$; $0^0=1$. Similarly, define B_i $(i=1, 2, \cdots, k)$ by $b_{pq}{}^{(i)}=\rho_i{}^{p-1}p^{q-1}$. Using the elementary properties of determinants, the author proves that $\det(A_1, A_2, \cdots, A_k)=\det(B_1, B_2, \cdots, B_k)=$

$$\prod_{j=1}^{k}\prod_{t=1}^{m_j}(m_j-t)!\prod_{j=1}^{k}\rho_j{}^{\frac{1}{2}m_j(m_j-1)}\prod_{\lambda=1}^{k-1}\prod_{i=\lambda+1}^{k}(\rho_i-\rho_\lambda)^{m_im_\lambda}.$$

R. G. Stanton (Waterloo, Ont.).

See also: Combinatorial Analysis: Ryser. **Numerical Methods:** Brauer.

Partial Order, Lattices

Gulotta, Beniamino. Reticoli e probabilità. Archimede **9** (1957), 145–159.

An expository treatment of lattices and of probability on lattices.

H. Rubin (Eugene, Ore.).

Felscher, Walter. Ein unsymmetrisches Assoziativ-Gesetz in der Verbandstheorie. Arch. Math. **8** (1957), 171–174.

The author demonstrates the equivalence and irreducibility of a number of axiom systems for lattices.

R. M. Baer (Berkeley, Calif.).

Ellis, David. Notes on the foundations of lattice theory. III. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 94–103.

[Part I appeared in Publ. Math. Debrecen **1** (1950), 205–208; MR **12**, 472; part II in Arch. Math. **4** (1953), 257–260; MR **15**, 192.] The author determines the implications and non-implications between several hypotheses relating the operations in a lattice L and an additional binary operation $*$: multiplicative lattices, delta-lattices, and various types of groupoids such as a C-groupoid, that is, with $x*y=(x\cap y)*(x\cup y)$. If L is a group (and still more if the group is abelian) under $*$ then some of the hypotheses are shown equivalent. Alternate forms are given for the hypotheses of B. H. Arnold [Pacific J. Math. **1** (1951), 33–41; MR **13**, 525] on $*$ in a distributive lattice. If L is a groupoid under $*$, it becomes a C-groupoid under $\dagger$ where $x\dagger y\equiv(x\cap y)*(x\cup y)$ is called the C-resolvent. The set of groupoids in L is a lattice L^*, mapped homomorphically onto the lattice of C-groupoids by the introduction of $\dagger$. Distributivity of a lattice M is shown equivalent to the existence of a superlattice of M which admits various kinds of quasi-groups.

P. M. Whitman (Baltimore, Md.).

Amemiya, Ichiro. On the representation of complemented modular lattices. J. Math. Soc. Japan **9** (1957), 263–279.

A new proof is given of von Neumann's theorem that every complemented modular lattice L which possesses an n frame with $n\geqq 4$ is isomorphic to the lattice of all principal left ideals of the ring of all $n\times n$ matrices over a regular ring. Several authors have offered simplifications of the original proof, but like von Neumann they all construct the ring by a generalization of von Staudt's algebra of throws. In the paper under review the main ideas are closer to Artin's method using the group of translations [Rep. Math. Colloq. (2) **2** (1940), 15–20; MR **3**, 179]. The fundamental lemma which guarantees the existence of sufficiently many "translations" is: Suppose p, q, c are independent elements contained in an element s of L, such that both p and q are perspective to a complement of s. If a and b are any complements of s such that $a\cup c=b\cup c$, then there exists one and only one automorphism f of L such that $f(a)=b$ and $f(x)=x$ whenever $x\leqq s$ or $c\leqq x$.

B. Jónsson (Minneapolis, Minn.).

Grätzer, G.; and Schmidt, E. T. On a problem of M. H. Stone. Acta Math. Acad. Sci. Hungar. **8** (1957), 455–460.

The authors define a pseudo-complemented distributive lattice to be a Stone lattice if $a^*\cup a^{**}=1$ for all elements of the lattice. They prove that a pseudo-complemented distributive lattice is a Stone lattice if and only if every two distinct minimal prime ideals are co-prime. It is further shown that a relatively pseudo-complemented distributive lattice is relatively Stone if and only if each non-comparable pair of prime ideals is co-prime.

R. P. Dilworth (Pasadena, Calif.).

Dwinger, Ph. Complete homomorphisms of complete lattices. Nederl. Akad. Wetensch. Proc. Ser. A. **60**= Indag. Math. **19** (1957), 412–420.

Some elementary properties of complete homomorphisms of complete lattices are derived and the results applied to extend somewhat known theorems on relatively complemented lattices and Boolean algebras.

R. P. Dilworth (Pasadena, Calif.).

Haupt, Otto; und Pauc, Christian Y. Über Adjunktion von Idealen in Booleschen Verbänden. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1957**, 177–193.

Die Verallgemeinerung der Theorie des Jordanschen Inhaltes und Lebesgueschen Masses von euklidischen Räumen auf lokal-kompakte Räume haben Verfasser [dieselben Abh. **1955**, 187–218; MR **17**, 469] auf rein verbandstheoretische Fragen (Adjunktion eines σ-Ideals neuer Nullmengen zu den Borelschen Nullmengen) geführt, die hier für sich behandelt werden. $a, b, c, \cdots$ seien Untermengen eines Booleschen Verbandes g. Enthält a mit je endlich bzw. abzählbar vielen Elementen auch ihre Vereinigung (ihren Durchschnitt), so heisst a ein s- (d-) bzw. σ- (δ-) System. $a \wedge b$ bzw. $a \vee b$ bzw. $a+b$ bezeichnet die Menge der Elemente $A \wedge B$ bzw. $A \vee B$ bzw. $A+B$ mit $A \in a$ und $B \in b$. c heisst ein Ideal bzw. σ-Ideal in g, wenn c ein s- bzw. σ-System ist mit $c \wedge g \subseteq c$. Zu einer Untermenge a wird nun ein (σ-) Ideal c adjungiert, d.h. $a \vee c$ und $a+c$ gebildet, und untersucht, wie sich die Eigenschaften von a vererben. Verfasser finden u.a., dass $a \vee c$ und $a+c$ mit a s- bzw. d-Systeme (σ- bzw. δ-Systeme) sind. $a+c$ ist mit a auch ein Boolescher (Boolescher σ-) Verband bzw. ein Ideal (σ-Ideal). Sind a und $a \vee c$ Boolesche Verbände, so gilt $a \vee c = a+c$.

P. Lorenzen.

See also: Foundations, Theory of Sets, Logic: Bruns. **Computing Machines:** Culbertson.

Fields, Rings

Michler, Lothar. Über eine Verallgemeinerung des Hauptsatzes der Galoisschen Theorie. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **1** (1956/57), 3–7.

Michler, Lothar. Über die Bedeutung des Faktorgruppoids $\Gamma \| \mathfrak{H}$ in einer verallgemeinerten Galoisschen Theorie. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **1** (1957), 77–80.

Let K be a separable non-normal extension of finite degree over the field k. Let Σ be the set of all algebraic extensions of k which are k-isomorphic to K. The set of all k-isomorphisms of K into fields in Σ forms a Mischgruppe in the sense of Loewy [J. Reine Angew. Math. **157** (1927), 239–254]. The set of all isomorphisms of one field in Σ into itself or another field in Σ is a Gruppoid in the sense of Brandt [Math. Ann. **96** (1926), 360–366]. Loewy has somewhat generalized Galois theory by using Mischgruppen and non-normal extensions [S.B. Heidelberger Akad. Wiss. **1925**, no. 7; **1927**, no. 1]. The present author gives a similar generalization using Gruppoide. His methods depend on symmetric functions and generation of a field by a single element.

G. Whaples.

Barnes, W. E.; and Schneider, H. The group membership of a polynomial in an element algebraic over a field. Arch. Math. **8** (1957), 166–168.

Let F be a field and R an extension ring of F. An element $a \in R$ is said to be a group-element in R if it is contained in some multiplicative subgroup of R. Now suppose that a is algebraic over F, and denote its minimum polynomial by $m(x)$. The two principal results proved in the present paper are as follows. (i) If $q(x)$ is a polynomial in $F[x]$, then $q(a)$ is a group-element in R if and only if $(q(x)^2, m(x)) = (q(x), m(x))$; if this condition is satisfied, then $q(a)$ is even a group-element in $F[a]$. (ii) $q(a)$ is a group-element in R for every polynomial $q(x) \in F[x]$ if and only if the irreducible factors of $m(x)$ are simple.

L. Mirsky (Sheffield).

Szász, F. A. Note on rings in which every proper left-ideal is cyclic. Fund. Math. **44** (1957), 330–332.

A ring is called cyclic if its additive group is a cyclic group, and a ring is called a ring with property P if each of its proper left ideals is a cyclic ring. The author proves that a ring has property P if and only if it is one of the following: a sfield, a cyclic ring, a zero-ring of type p^∞, or a ring of order p^2 (where p is a prime).

W. E. Deskins (East Lansing, Mich.).

Kertész, Andor. Semisimple rings as operator domains. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 149–186. (Hungarian)

A systematic exposition of the results published by the author in two articles in English [Publ. Math. Debrecen **3** (1954), 289–296 (1955); **4** (1955), 79–86; MR **17**, 342; **16**, 991.]

Kertész, A. Systems of equations over modules. Acta Sci. Math. Szeged **18** (1957), 207–234.

The problem of solving a system of linear equations over an R-module G is shown to be equivalent to the problem of extending a homomorphism of some module M into G to a module $N \supset M$ into G. Therefore, if $\hat{G}$ is the minimal injective extension of G, any compatible system of linear equations over G can be solved in $\hat{G}$. If $\hat{G}=G$, then every compatible system of equations over G can be solved in G. This is the case, for example, if the ring R is semi-simple in the classical sense and the module G has no R-annihilating submodule. Much of the paper is devoted to well-known properties of injective modules.

R. E. Johnson (Northampton, Mass.).

Henriksen, Melvin. Some remarks on elementary divisor rings. II. Michigan Math. J. **3** (1955–1956), 159–163.

This paper continues the investigations of Helmer [Bull. Amer. Math. Soc. **49** (1943), 225–236; MR **4**, 185], Kaplansky [Trans. Amer. Math. Soc. **66** (1949), 464–491; MR **11**, 155], and the author jointly with the reviewer [ibid. **82** (1956), 362–365, 366–391; MR **18**, 9]. A commutative ring S with identity is called an elementary divisor (resp. Hermite) ring if for every matrix A over S there exist nonsingular P, Q such that PAQ (resp. AQ) is diagonal (resp. triangular). The main results are: (1) If all finitely generated ideals in S are principal, and if the Perlis-Jacobson radical R contains a prime ideal of S then S is an Hermite ring. (2) An Hermite ring S is an elementary divisor ring if and only if S/R is an elementary divisor ring. (3) If S is an Hermite ring and every element not in R is contained in only finitely many maximal ideals, then S is an elementary divisor ring Next, an ex-

ample is given of an integral domain that is an elementary divisor ring but is not adequate (in the sense of Helmer). Finally, some open problems are stated. *L. Gillman.*

Gillman, L. Rings with Hausdorff structure space. Fund. Math. **45** (1957), 1–16.

The paper is related to previous results of M. Henriksen and the author [Trans. Amer. Math. Soc. **82** (1956), 366–391; MR **18**, 9; and earlier papers]. Let A be a ring, and $\mathfrak{S}$ a set of two-sided proper ideals in A; $\mathfrak{S}$ is called a structure set of A if the relation $I \cap JCS$ implies ICS or JCS, with $S \in \mathfrak{S}$, and I, J intersections of ideals in $\mathfrak{S}$. When equipped with the Stone topology, $\mathfrak{S}$ is called a structure space of A; it is necessarily T_0 but not in general T_1. After some preliminaries, a characterisation of rings with Hausdorff structure space is obtained. The cases $\mathfrak{S}$=all primitive ideals in A, $\mathfrak{S}$=all proper prime ideals in A are given prominence; various theorems are proved relating the properties of $\mathfrak{S}$ to properties of ideals in A. Conditions for various types of regularity are obtained. In the final section commutative ordered rings are treated. {Editor's note: the author writes that 4.1 is false, M. Henriksen and D. G. Johnson having supplied a counterexample; but the later results are not affected, as pointed out by M. Jerison.} *J. H. Williamson*

Fröhlich, A. Distributively generated near-rings. I. Ideal theory. II. Representation theory. Proc. London Math. Soc. (3) **8** (1958), 76–94, 95–108.

The axiomatization of algebraic systems generated by group endomorphisms leads to the study of distributively generated near-rings (DGN-rings). The author lists a number of examples of DGN-rings, then proceeds with an investigation of analogues and generalizations of many of the standard concepts of ring theory: homomorphisms, ideals and residue DGN-rings, one- and two-sided ideals and modules (not to be confused with ring modules) and their respective lattices, nilpotency, representation by mappings of groups, and idempotents and related decompositions. The following illustrates the type of result obtained: Let g be an idempotent of R. Then $l(g)$, the left ideal of left annihilators of g, is complemented in R^+ by the left R-module Rg, and g is a right identity element in Rg. If in addition g is a distributive element, then $r(g)$, the set of right annihilators of g in R, is a right ideal and gR is a right R-module in R, with left identity g; $gR+r(g)=R^+$ and $gR \cap r(g)=(0)$.

Also some concepts peculiar to DGN-rings are discussed. If x, a and b are elements of R, then $x(a+b)-xb-xa$ is defined to be the distributor $[x|a,b]$ of a, b for x. The distributor is shown to have properties analogous to those of the commutator in group theory. Moreover the author demonstrates that it is closely related to the commutator of the additive group of the DGN-ring.

{It should be noted that another approach to the axiomatization of algebraic systems generated by group endomorphisms has been developed by H. Fitting [Math. Ann. **107** (1932), 514–542].} *W. E. Deskins.*

See also: Partial Order, Lattices: Amemiya. **Groups and Generalizations:** Thierrin. **Banach Spaces, Banach Algebras, Hilbert Spaces:** Aurora.

Algebras

Yoshii, Tensho. Note on algebras of strongly unbounded representation type. II. Proc. Japan Acad. **32** (1956), 744–747.

Continuation of same Proc. **32** (1956), 383–387 [MR **18**, 108], giving one further sufficient condition for an algebra to be of strongly unbounded representation type; it is stated that as a result a conjecture of R. Brauer and R. M. Thrall, that unbounded and strongly unbounded representation types coincide, holds in a special case.

Kostrikin, A. I.; and Šafarevič, I. R. Groups of homologies of nilpotent algebras. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 1066–1069. (Russian)

Let the nilpotent associative algebra N of finite rank over the field k have Betti numbers b_n and let $R_N(t)=\sum_{n=0}^{\infty} b_n t^n$ be the associated Poincaré function. If N is the direct sum of m nilpotent algebras $N_1+\cdots+N_m$, then $[R_N(t)]^{-1}-1=\sum_{i=1}^{m}\{[R_{N_i}(t)]^{-1}-1\}$. If the notation $F(t)\gg G(t)$ is used to mean that each coefficient in the power series $G(t)$ does not exceed the corresponding coefficient in the power series $F(t)$, then $(1+t)^{-1}R_N(t)\gg(1-t^2)^{-1}$ and hence each Betti number is positive. On the other hand, if N has rank r, then $R_N(t)\ll(1-rt)^{-1}$. If the set of Betti numbers for N is bounded and if k is a finite field, then $R_N(t)$ is a rational function. *R. A. Good.*

Jacobson, N. Generation of separable and central simple algebras. J. Math. Pures Appl. (9) **36** (1957), 217–227.

An algebra C is a Frobenius algebra if there exists a hyperplane in C which contains no non-zero one-sided ideals of C. Let A be a central simple algebra of order n^2 over its centre Φ. The author considers the Frobenius subalgebras C of A for which $(C:\Phi)=n$. These subalgebras contain the maximal subfields of A and, in fact, it is shown that they possess most of the properties of the maximal subfields. This follows from the fact that A considered as a C-bimodule — i.e., C operates on A both by right and left multiplications — is cyclic. That is, there exists $y \in A$ such that $A=CyC$. The following is then readily obtained: 1) (Albert-Kasch) If A is a separable semi-simple algebra over an infinite field Φ, then A contains an element x and a conjugate x' of x such that $A=(\Phi[x])(\Phi[x'])$ [e.g., F. Kasch, J. Reine Angew. Math. **189** (1951), 150–159; MR **14**, 239]; 2) C is its own centralizer in A; 3) every isomorphism (derivation; higher derivation) of C into A can be extended into an inner automorphism (derivation, etc.) of A. The paper concludes by showing that the preceding result yields the usual cross product representation of A with certain Frobenius subalgebras replacing the maximal normal fields. A similar result is obtained for algebras of characteristic $p\neq 0$, where derivations replace the automorphisms.

S. A. Amitsur (Notre Dame, Ind.).

Albert, A. A. On partially stable algebras. Trans. Amer. Math. Soc. **84** (1957), 430–443.

Let A be a simple commutative power-associative algebra of degree t over a field F of characteristic $\neq 2, 3, 5$. If $t\geqq 3$, or if $t=2$ and the characteristic of F is 0, then it is known that A is a Jordan algebra [Albert, same Trans. **69** (1950), 503–527; MR **12**, 475; Kokoris, Ann. of Math. (2) **64** (1956), 544–550; MR **18**, 375]. On the other hand, if $t=2$ and the characteristic of F is $p>5$, examples are known of algebras A which are not Jordan algebras [Kokoris, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 534–537; MR **14**, 11; Albert, Trans. Amer. Math. Soc. **74** (1953), 323–343; MR **14**, 614]. In this paper the author characterizes those simple commutative power-associative algebras A of degree two over an algebraically closed F of characteristic $p>5$ which are partially stable in the following sense: A contains a (stable) idempotent $u\neq 1$ which

is such that $A_u(1)A_u(\frac{1}{2})\subseteq A_u(\frac{1}{2})$, $A_u(0)A_u(\frac{1}{2})\subseteq A_u(\frac{1}{2})$, where $A_u(\lambda)$ consists of all elements x of A satisfying $ux=\lambda x$.

Let B be a commutative associative algebra with unity element 1 over F of characteristic $p>5$, and let $B=1F+N$ for N the radical of B. Let L be the sum (but not necessarily vector space direct sum) of $m+1$ homomorphic images y_iB of B, and let T be the vector space direct sum $T=B+L$. Let D be a set of $(m+1)^2$ derivations D_{ij} of B subject to the conditions $D_{ij}=-D_{ji}$ $(i, j=0, \cdots, m)$. Define multiplication in T by $y_0^2=1$, $y_0y_j=0$, $y_iy_j=b_{ij}=b_{ji}$ $(i, j=1, \cdots, m)$, where the b_{ij} are arbitrary elements of B, and define $(y_ia)b = b(y_ia)=y_i(ab)$, $(y_ia)(y_jb)=b_{ij}(ab)+(aD_{ij})b-a(bD_{ij})$. Then T is a commutative power-associative algebra which may be imbedded in a commutative algebra A which is the vector space direct sum $A=T+zB$, where multiplication is defined by $(za)b=z(ab)$, $(za)(zb)=ab$ for every a and b of B, and $(zB)L=0$. This algebra A is power-associative, and is partially stable, with $2u=1+z$. The author shows that A is simple if and only if B is D-simple and there exists no g in L such that $gx=0$ for every x in L. The principal theorem of the paper is the converse that, if A is a simple commutative power-associative algebra of degree two which is partially stable (for an idempotent u) over an algebraically closed field F of characteristic $p>5$, then $A_u(1)$ is associative and A has the form $A=T+zB=B+zB+L$ described above with $B\cong A_u(1)$.

{Reviewer's note: L. A. Kokoris has pointed out to the author that there is an error in the proof of the principal result. It occurs on line 8 of page 434, and invalidates a number of the subsequent arguments. However, the results of the paper are correct, and new proofs will appear in an addendum [see below].} *R. D. Schafer.*

Albert, A. A. Addendum to the paper on partially stable algebras. Trans. Amer. Math. Soc. **87** (1958), 57–62.

A revised proof of the principal theorem of the paper reviewed above is given. See the final paragraph of the review above. *R. D. Schafer* (Princeton, N.J.).

Jacobson, N.; and Paige, L. J. On Jordan algebras with two generators. J. Math. Mech. **6** (1957), 895–906.

It is proved that the free Jordan algebra with two generators whose characteristic is not two is special. This result was obtained earlier by A. I. Shirsov [Mat. Sb. N.S. **38(80)** (1956), 149–166; MR **17**, 822], but the proof here is simpler and also gives a simple basis for the algebra. The proof is made by first considering the free special Jordan algebra $\mathfrak{J}_0$ with two generators. This is the subalgebra relative to Jordan multiplication generated by two generators in a free associative algebra. A basis S_0 is defined inductively for $\mathfrak{J}_0$. This procedure is then imitated in the free Jordan algebra $\mathfrak{J}$ with two generators to obtain a basis S for $\mathfrak{J}$ which is in 1-1 correspondence with S_0. A theorem of P. Cohn, together with Shirsov's theorem, implies that any Jordan algebra with two generators whose characteristic is not two is special.

L. A. Kokoris (Chicago, Ill.).

Weiner, L. M. Lie admissible algebras. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 10–24.

From any linear algebra A (of finite dimension, over a field of characteristic $\neq 2$) two algebras $A^{(-)}$, $A^{(+)}$ may be obtained by defining on the space of A the multiplications $[a, b]=\frac{1}{2}(ab-ba)$, $\{a, b\}=\frac{1}{2}(ab+ba)$ respectively. The algebra A is called Lie admissible if $A^{(-)}$ is a Lie algebra, L-simple if $A^{(-)}$ is simple; and Jordan admissible, J-simple are defined similarly in terms of $A^{(+)}$. The author proves the following decomposition theorem: Let A be a flexible power associative algebra of characteristic 0 such that $B=A^{(-)}$ is a direct sum of simple Lie algebras $B_i=A_i^{(-)}$, where the A_i are subspaces of A. Then the A_i are ideals in A. Without the assumption of flexibility this result is false, as is shown by the case of algebras of degree 1, i.e., algebras A in which $x^2=f(x)x$, where the scalar function $f(x)$ is necessarily linear. The multiplication of such algebras is determined by f and the multiplication in $A^{(-)}$. Let L be a simple Lie algebra (multiplication $[x, y]$) and f any linear function on L. Then the algebra consisting of the space L with the multiplication $xy=\frac{1}{2}(f(x)y+f(y)x)+[x, y]$ is a simple power associative algebra of degree 1, distinct from the known types. Finally, Lie admissible J-simple algebras are considered. Albert [Summa Brasil. Math. **2** (1951), 183–194; MR **14**, 11] has shown that if $A^{(+)}$ is a central simple J-algebra of $t\times t$ matrices $(t>2)$, then for some scalar extension K, the multiplication in A_K is $x\cdot y=\frac{1}{2}(xy+yx)+(xy-yx)T$, for a certain linear transformation T. The author proves that if $A^{(+)}$ is isomorphic to the algebra of all $t\times t$ matrices $(t>2)$, $A^{(-)}$ is a Lie algebra, and T is non-singular on the space spanned by the $xy-yx$ $(x, y\in A)$, then T has the form $aT=\alpha a+\lambda(a)1$, where $\alpha\in K$, λ is a linear function and a is any matrix of trace zero. But further examples show that general Lie admissible J-simple algebras allow a more complicated form of T. *P. M. Cohn.*

Oehmke, Robert H. A class of noncommutative power-associative algebras. Trans. Amer. Math. Soc. **87** (1958), 226–236.

The class of nonassociative algebras A satisfying $x(xx)=(xx)x$ and having $A^{(+)}$ a separable Jordan algebra is called the class SJ. Let G be a nonassociative algebra over a field F of characteristic not 2 which contains a subspace S closed under the operation $(x, y)=\frac{1}{2}(xy+yx)$, where xy is the product in G. Also let T be any linear mapping on the subspace $U(S)$ generated by elements of the form $xy-yx$ with x, y in S into S. An algebra $B(G, S, T)$ which is the same vector space as S is defined by the product $x\cdot y=\frac{1}{2}(xy+yx)+(xy-yx)T$. Then $B(G, S, T)$ is said to be bonded to G. If an algebra A over a field F is of class SJ and is such that $A^{(+)}$ is special, there exists an associative algebra G over F such that $A=B(G, S, T)$. But if $A^{(+)}$ has an ideal I that is a simple, separable, exceptional Jordan algebra, then I is an ideal of A and the multiplication of I in A agrees with the multiplication of I in $A^{(+)}$. These results combine to give the fact that for any algebra of class SJ, there exists an algebra G that is a direct sum of a semisimple associative algebra and algebras of three-rowed matrices over a Cayley algebra such that $A=B(G, S, T)$. A class of algebras A satisfying $x(xx)=(xx)x$ such that $A^{(+)}$ is a semisimple Jordan algebra with an inseparable component is exhibited for which no bonding mapping T exists. The following application is proved: if A is a finite power-associative ring without elements of additive order 2, 3 or 5; and if for every a in A, $a^{n(a)}=a$, $n(a)$ an integer greater than 1; then A is a vector space sum $A=A_1+A_2$; the subspace A_1 is an ideal of A and is a direct sum of finite fields; the subspace A_2 is bonded to A, and $A_2^{(+)}$ is a direct sum of classical 3-dimensional central simple Jordan algebras without nilpotent elements. This last result is an extension of a theorem of R. D. Schafer. *L. A. Kokoris.*

Groups and Generalizations

Wallace, D. A. R. Note on the radical of a group algebra. Proc. Cambridge Philos. Soc. **54** (1958), 128–130.

Let G be a group of order mp^a, $(m, p)=1$, p a prime, and let F be an algebraically closed field of characteristic p. The author proves that r, the dimension of the radical of the group algebra of G over F, is greater than or equal to p^a-1, and that $r=p^a-1$ if and only if any Sylow p-subgroup of G is its own normalizer while the intersection of any two distinct Sylow p-subgroups of G is the unit element. *W. E. Deskins* (East Lansing, Mich.).

★ **Novikov, P. S. On the algorithmic insolvability of the word problem in group theory.** American Mathematical Society Translations, Ser. 2, Vol. 9, pp. 1–122. American Mathematical Society, Providence, R. I., 1958. iii+231 pp. $5.00.

Translated by K. A. Hirsch from Trudy Mat. Inst. im. Steklov. no. **44**. Izdat. Akad. Nauk SSSR, Moscow, 1955; MR **17**, 706.

Dlab, Vlastimil. D-Rang einer Abelschen Gruppe. Časopis Pěst. Mat. **82** (1957), 314–334. (Czech. Russian and German summaries)

Mit G wird immer eine abelsche (additiv geschriebene) Gruppe bezeichnet. Eine Untermenge $(g_\iota)_{\iota\in I}$ von G heisst D-unabhängig, wenn jede Beziehung $k_1\cdot g_{\iota_1}+k_2\cdot g_{\iota_2}+\cdots+k_n\cdot g_{\iota_n}=0$ mit ganzen k_i und beliebigen natürlichen n $k_i\cdot g_{\iota_i}=0$ für alle $i=1, 2, \cdots, n$ nach sich zieht. Ein maximales D-un- abhängiges System (von Null verschiedener Elemente) wird ein D-System genannt. Dessen Mächtigkeit ist im allgemeinen kein Invariant der Gruppe. Aber speziell in einer p-primären Gruppe $G_{(p)}$ haben alle D-Systeme dieselbe Mächtigkeit, die der Autor den D-Rang $r_D(G_{(p)})$ der Gruppe $G_{(p)}$ nennt. Der D-Rang $r_D(G)$ einer beliebigen (abelschen) Gruppe G wird durch die Beziehung $r_D(G)=r(G)+\sum_p r_D(P_{(p)})$ definiert, wo $r(G)$ den (gewöhnlichen) Rang der Gruppe G und $P=\sum'_p P_{(p)}$ die direkte Zerlegung der maximalen periodischen Untergruppe in G in p-primären Komponenten bedeutet.

Ist $\mathfrak{G}$ ein D-system der Gruppe G, $m(\mathfrak{G})$ seine Mächtigkeit, dann gilt $m(\mathfrak{G})\leqq r_D(G)$; ist darüber $r_D(G)\geqq\aleph_0$, so gilt $m(\mathfrak{G})=r_D(G)=m(G)$. Der Autor beweist folgende tiefere Behauptung: Ist $\mathfrak{M}=(\mathfrak{G}_\delta)_{\delta\in\Delta}$ das System aller D-Systeme einer Gruppe G ($\neq 0$), $m(\mathfrak{G}_\delta)=\lambda_\delta$ für $\delta\in\Delta$, dann hat die Menge der Kardinalzahlen $\Lambda=(\lambda_\delta)_{\delta\in\Delta}$ ein maximales Element λ_{δ_1} und ist $\lambda_{\delta_1}=r_D(G)$. Die Menge Λ enthält genau ein Element $\Lambda=(r_D(G))$, wenn und nur wenn in $\mathfrak{M}$ mindestens ein unendliches D-System existiert oder wenn die maximale periodische Untergruppe in G p-primär oder gleich Null ist.

Für jede Untergruppe H in G gilt (1) $r_D(H)\leqq r_D(G)\leqq r_D(H)+r_D(G/H)$. Ist $r_D(G)\geqq\aleph_0$ (oder $m(G)>\aleph_0$), dann gilt die Gleichheit in (1). Im Falle des endlichen D-Ranges $r_D(G)$ gilt die Gleichheit in (1), wenn und nur wenn H die schwache Servanzuntergruppe ist (d.h. wenn jede Restklasse in G/H von der Primzahlordnung ein Element von derselben Ordnung enthält).

Ein kanonisches D-System ist ein solches D-System, dessen jedes Element eine unendliche oder eine Primzahlpotenzordnung hat. Ist $\mathfrak{G}$ ein kanonisches D-System der Gruppe G, dann gilt die Beziehung $m(\mathfrak{G})=r_D(G)$. Im Falle, dass $r_D(G)$ endlich ist, sind kanonische D-Systeme genau die D-unabhängigen Mengen $\mathfrak{G}$, für die $m(\mathfrak{G})=r_D(G)$ gilt.

Eine periodische Gruppe G vom endlichen D-Range n ist eine direkte Summe von n Gruppen vom D-Range 1. *F. Šik.*

Berman, S. D. Generalized characters of finite groups. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 112–115. (Ukrainian. Russian and English summaries)

Let Φ be a group of such transformations of a finite group G over itself that if $\varphi\in\Phi$, then φ carries each character χ of the group G into a character and every class of conjugate elements of G into a class.

The author studied the generalized characters of the group G, each of which is obtained from some character χ by adding different characters of the form $\chi[\varphi(g)]$ ($g\in G$), where φ ranges over Φ. *Author's summary.*

Boone, William W. An analysis of Turing's "The word problem in semi-groups with cancellation". Ann. of Math. (2) **67** (1958), 195–202.

A careful review of a paper by Turing [Ann. of Math. (2) **52** (1950), 491–505; MR **12**, 239]. Arguments are given sustaining the proofs of some half dozen results of Turing. *R. M. Baer* (Berkeley, Calif.).

Thierrin, Gabriel. Contribution à la théorie des anneaux et des demi-groupes. Comment. Math. Helv. **32** (1957), 93–112.

Soit D un demi-groupe ou un anneau, H une partie non vide de D. Le complexe H est réflectif, si la relation $ab\in H$ ($a, b\in D$) entraîne $ba\in H$. Les idéaux réflectifs, le radical réflectif d'un idéal réflectif, les anneaux et les demi-groupes dont tous les idéaux à droite sont réflectifs sont étudiés.

Un complexe H est dit complètement réflectif, s'il est réflectif et si la relation $abc\in H$ entraîne $cba\in H$. En particulier les décompositions d'un idéal comme intersection d'idéaux complètement réflectifs primaires sont étudiées.

On trouve dans le travail des définitions diverses de la sorte citée. Les relations entre les notions introduites sont étudiées. Le travail contient 42 théorèmes. *Š. Schwarz* (Bratislava).

Tamura, T. Errata: The theory of construction of finite semigroups. I. Osaka Math. J. **9** (1957), 241.

The relevant article is in same J. **8** (1956), 243–261; MR **18**, 717.

Iséki, Kiyoshi. Contributions to the theory of semigroups. VI. Proc. Japan Acad. **33** (1957), 29–30.

[For parts I–V see same Proc. **32** (1956), 174–175, 225–227, 323–324, 430–435, 560–561; MR **17**, 1184; **18**, 282, 872.] By a character of a semigroup S we mean a complex-valued function $\chi(x)$ satisfying $\chi(ab)=\chi(a)\chi(b)$ for all $a, b\in S$. Let $\hat{S}$ denote the semigroup of all characters (with the natural multiplication in $\hat{S}$). If $\mathfrak{a}$ is an ideal of S, then the set $\hat{\mathfrak{a}}$ of all $\chi\in\hat{S}$ with $\chi(x)=0$ for all $x\in\mathfrak{a}$ is an ideal of $\hat{S}$. It is proved: If S is a commutative periodic semigroup with a finite number of idempotents, there is a one-to-one correspondence between the closed ideals of S and the ideals of $\hat{S}$, and $\mathfrak{a}\subset\mathfrak{b}\Leftrightarrow\hat{\mathfrak{a}}\supset\hat{\mathfrak{b}}$. Here the term "closed ideal" is used in the following sense: $\mathfrak{a}$ is closed if $x^\rho\in\mathfrak{a}$ with an integer $\rho>0$ implies $x\in\mathfrak{a}$. This is a slight generalization of a result of the reviewer, who considered the case of finite semigroups [Czechoslovak Math. J. **4(79)** (1954), 296–313; MR **16**, 186].

{Reviewer's note: Using some results of another paper

of the reviewer [ibid. **6(81)** (1956), 330–364; MR 1063], the results of the present paper and of the reviewer can be generalized to a great extent to the case of Hausdorff compact semigroups. Related results are also announced in a preliminary report by N. J. Rothman [Bull. Amer. Math. Soc. **63** (1957), 360].} *S. Schwarz* (Bratislava).

Ikuta, Toshiharu. A remark on a loop. Nat. Sci. Rep. Lib. Arts Sci. Fac. Shizuoka Univ. **2** (1957), no. 1, 1–4.

A multiplicative loop, in which, for every element x, xT is defined by $x\cdot xT=1$, is said to have the property (π) (also called the "crossed-inverse property") if the identity $xy\cdot xT=y$ holds. A multiplicative loop is said to have the property (*) if it satisfies the identity $(xy)T=xT\cdot yT$. (π) implies (*), but (π) is not a consequence of (*). If xT^{n+1} is defined as $(xT^n)T$, and if k is the smallest positive integer satisfying $xT^k=x$, then the elements xT^n $(n=1, 2, \cdots, k)$ are called an inverse-cycle of length k. The reviewer has shown [Proc. Amer. Math. Soc. **6** (1955), 448–453; MR **16**, 1083] that a loop with the property (π) cannot consist only of the unit element and m inverse-cycles of equal length n, unless n divides $2m$. In the present paper the author shows that this statement is true also when (π) is replaced by the weaker requirement (*).

{An equivalent result has been stated earlier by the reviewer [Bull. Amer. Math. Soc. **63** (1957), 92].}

R. Artzy (Haifa).

Wall, Drury W. Sub-quasigroups of finite quasigroups. Pacific J. Math. **7** (1957), 1711–1714.

If a quasigroup Q of finite order n has a proper subquasigroup of order s, then $n\geqq 2s$. If Q has two proper subquasigroups R, S of orders r, s, with intersection $P=R\cap S$ of order p (necessarily a subquasigroup if non-null), then

$$(*) \qquad n\geqq r+s+\max(r, s)-2p.$$

If equality occurs here, then $r=s$ if and only if $P\cup W$ is a subquasigroup, where W is the set-theoretic complement of $R\cup S$ in Q. Examples are given showing equality in (*) in both cases $r=s$, $r\neq s$. When $r=s$ and P is null, (*) becomes $n\geqq 3s$ and W is a subquasigroup. So one might conjecture that existence of three mutually disjoint subquasigroups of Q all of order s would imply that $n\geqq 4s$, and perhaps that the remaining elements would form a fourth subquasigroup; the author states that both conjectures are false. *I. M. H. Etherington* (Edinburgh).

See also: Fields, Rings: Fröhlich. **Lie Groups and Algebras:** Mostow. **Algebraic Geometry:** Borel.

Homological Algebra

★ **Ikeda, Masatoshi; Nagao, Hiroshi; and Nakayama, Tadasi. Cohomology theory for algebras.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 235–238. Science Council of Japan, Tokyo, 1956.

The results of this short note are contained in the longer paper by the same authors in Nagoya Math. J. **7** (1954), 115–131 [MR **16**, 214]. *J. Tate.*

See also: Linear Algebra: Kostant. **Algebras:** Kostrikin and Šafarevič.

THEORY OF NUMBERS

General Theory of Numbers

★ **Landau, Edmund. Elementary number theory.** Translated by J. E. Goodman. With exercises by P. T. Bateman and E. E. Kohlbecker. Chelsea Publishing Co., New York, N.Y., 1958. 256 pp. (1 insert) $4.95.

A translation from the author's well-known "Elementare Zahlentheorie", which is Vol. 1 of "Vorlesungen über Zahlentheorie" [Leipzig, 1927].

Yasinovyĭ, È. A. Irrationality of certain values of trigonometric functions. Mat. v Škole 1958, no. 3, 3–6. (Russian)

Simple elementary proofs of the irrationality of certain numbers; in particular, $\cos \pi/n$; $n>3$ and $\sin \pi/n$; $n\neq 1, 2, 6$.

Prakash Gupta, Om. Partitions into exactly k distinct primes. Res. Bull. Panjab Univ. no. **107** (1957), 283–290.

The author gives the values of $R_k(n)$, the number of partitions of n into exactly k distinct primes, for $1\leqq k\leqq n\leqq 300$. The method of computation is based on the observations that if $R_k(n, p)$ is the number of partitions of n into exactly k distinct primes the smallest of which is p, then

$$R_k(n)=\sum_p R_k(n, p),\quad R_k(n, p_0)=\sum_{p>p_0} R_{k-1}(n-p_0, p).$$

M. Newman (Washington, D.C.).

Skolem, Th. On certain distributions of integers in pairs with given differences. Math. Scand. **5** (1957), 57–68.

The first problem considered is whether the integers from 1 to $2n$ can be distributed into n pairs (a_r, b_r) so that $b_r-a_r=r$ for $r=1, \cdots, n$. It is proved that this is possible if $n\equiv 0$ or $1 \pmod 4$, but not otherwise. The analogous problem for all the positive integers is solved by taking $a_1=1$ and a_r as the least integer other than a_s, a_s+s for $s<r$. It is shown that this construction gives $a_r=[r\theta]$, where $\theta=\frac{1}{2}(1+\sqrt{5})$. The paper goes on to discuss the sequences $[r\alpha]$, where $\alpha>1$ is irrational and $r=1, 2, \cdots$. Some of the results obtained are not new [see Uspensky and Heaslet, Elementary number theory, McGraw-Hill, New York, 1937, p. 98; MR **1**, 38]. *H. Davenport.*

Bang, Thøger. On the sequence $[n\alpha]$, $n=1, 2, \cdots$. Supplementary note to the preceding paper by Th. Skolem. Math. Scand. **5** (1957), 69–76.

Further results are proved concerning the sequences of the title [see the preceding review]. In particular, the sequence $[n\alpha]$ contains the sequence $[n\gamma]$ if and only if there exist positive integers a, c such that $a(1-\alpha^{-1})+c\gamma^{-1}=1$. The asymptotic density of the common part of two sequences $[n\alpha]$, $[n\beta]$ is investigated. *H. Davenport.*

Skolem, Th. Über einige Eigenschaften der Zahlenmengen $[\alpha n+\beta]$ bei irrationalem α mit einleitenden Bemerkungen über einige kombinatorische Probleme. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 42–49.

This overlaps with the paper reviewed second above

but contains some extensions of results proved there. It is proved, for example, that the sets $[\alpha n+\gamma]$, $[\beta n+\delta]$ are disjoint if and only if there exist positive integers a, b such that $a\alpha^{-1}+b\beta^{-1}=1$, $a\gamma\alpha^{-1}+b\delta\beta^{-1}=$integer.

H. Davenport (Cambridge. England).

Erdös, P. Über eine Art von Lakunarität. Colloq. Math. **5** (1957), 6–7.

A sequence $n_1<n_2<\cdots$ of integers is said to be lacunary in the sense of S. Hartman if there exists no number k such that, for each integer r, an integer m can be found for which $n_{m+i+1}-n_{m+i}<k$ $(i=1, 2, \cdots, r)$. Hartman [Matematiche Catania **10** (1955), 57–61; MR **17**, 586] himself, after formulating this new type of gap condition, gave an example of a lacunary sequence having positive density, and shortly afterwards Sierpiński [Matematiche **10** (1955), 67–70; MR **17**, 586] proved that the sequence of primes is lacunary. The author has discovered a class of sequences which are readily tested for Hartman's gap condition; he has proved, in a remarkably brief and simple manner, the following general theorem: Let $1<a_1<a_2<\cdots$ be a sequence of integers. Let $A(a_1, a_2, \cdots)$ be the sequence of those integers which are not divisible by any single a_i. Then $A(a_1, a_2, \cdots)$ is lacunary if, and only if, there exists an infinite sub-sequence $a_{i_1}, a_{i_2}, \cdots$ satisfying $(a_{i_1}, a_{i_2})=1$. It follows in particular, on putting $a_i=p_i^2$, that the square-free integers are lacunary (yet possess positive density), and hence also that the primes are lacunary.

H. Halberstam (London).

Newman, Morris. Congruences for the coefficients of modular forms and some new congruences for the partition function. Canad. J. Math. **9** (1957), 549–552.

Let $p_r(n)$ be the coefficient of x^n in $\prod_{n=1}^{\infty}(1-x^n)^r$, so that $p_r(n)=p(n)$, the partition function, when $r=-1$. The author proves the following theorem. Let r be even, $4\leqq r\leqq 24$ and p a prime greater than 3 such that $\delta=r(p-1)/24$ is an integer. Then if Q, n are integers and $R=Qp+r$,

$$p_R(np+\delta)\equiv p_r(\delta)p_{Q+r}(n) \pmod{p}.$$

From this he deduces that, for $n\equiv 6 \pmod{13}$,

$$p(13^2n-7)\equiv 6p(n) \pmod{13}.$$

A corollary is that $p(a\Delta_n+b)\equiv 0 \pmod{13}$, for a variety of different integers a, b, where $\Delta_n=13(13^{2n}-1)/24$.

R. A. Rankin (Glasgow).

Mordell, L. J. Note on simultaneous quadratic congruences. Math. Scand. **5** (1957), 21–26.

With an odd prime p as the modulus, the author considers m simultaneous congruences

$$a_{r1}x_1^2+\cdots+a_{rn}x_n^2\equiv -a_{r0} \quad (r=1, \cdots, m).$$

The number of solutions is denoted by N. Under certain conditions of linear independence in the coefficient matrix, the author estimates $N-p^{n-m}$. He suggests that the best possible result will be obtained by application of the following conjecture: If p is a prime and if $B_r=b_{1r}t_1+\cdots+b_{kr}t_k$ $(r=0, 1, \cdots, h)$ are linear forms with integral coefficients, then the multiple sum of Legendre symbols $\sum\left(\frac{B_0B_1\cdots B_h}{p}\right)$ (summation is taken over complete residue sets mod p for $t_1, \cdots, t_k$) is $O(p^{k(k+1)})$ if h is odd, and zero if h is even. If $k=2$, the conjecture follows from a deep result by Weil [Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 345–347; MR **2**, 345]. From the case $k=2$ the author derives estimates for cases with $k>2$, but these are quite far from the conjectured results. {It should be noted that there are exceptional cases, both in the conjecture for $k>2$ as in Weil's case $k=2$. These exceptional cases are neglected in the paper, and in the applications made by the author it may not always be easy to decide whether these exceptions do or do not cause difficulties.}

The author does not state explicitly that his formulas for general values of m are restricted to the case $m>2$, but the case $m=2$ is dealt with in detail at the end of his paper, and for this case he gives a number of explicit formulas.

N. G. de Bruijn (Amsterdam).

Mordell, L. J. Corrigendum: On the four integer cubes problem. J. London Math. Soc. **32** (1957), 383.

The derivation in J. London Math. Soc. **11** (1936), 208–218, of sufficient conditions that $x^3+y^3+z^3+w^3=a^3+b^3+c^3+d^3$ (a, b, c, d given integers) have an infinity of integer solutions, is invalid if $a=b$, $c=d$, etc.

Wakulicz, A. On the equation $x^3+y^3=2z^3$. Colloq. Math. **5** (1957), 11–15.

A proof of the well-known theorem (first published by Euler in 1770 and included, with generalizations, in a number of standard undergraduate texts) that rational integral solutions of the equation in the title require $x=\pm y$.

J. D. Swift (Los Angeles, Calif.).

Lavrik, A. F. Representation of numbers as a sum composed of a prime and a power of a given integer. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 445–446. (Russian)

The author states five theorems. The first one is: in the interval $(0, x)$ there exist more than $\alpha x/\log a$ numbers representable in one and only one way as the sum of a prime and a power of the given integer $a>a_0$, where α is an absolute positive constant. The other four theorems are extensions. A proof is sketched for the third one.

Prachar, Karl. Über die kleinste quadratfreie Zahl einer arithmetischen Reihe. Monatsh. Math. **62** (1958), 173–176.

Let $Q(x, k, l)$ denote the number of integers $m\leqq x$ which are $\equiv l \pmod{k}$, and such that m is squarefree. It is proved that (with $(k, l)=1$)

$$Q(x, k, l)=Axk^{-1}+O(x^{\frac{1}{2}}k^{-\frac{1}{2}+\varepsilon}+k^{\frac{1}{2}+\varepsilon})$$

for any given $\varepsilon>0$, where

$$A=\frac{6}{\pi^2}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)^{-1},$$

and the O is uniform in k. From this it is concluded that, for $q_1(k, l)=$the smallest squarefree number in the arithmetic progression of integers $\equiv l \pmod{k}$, one has

$$q_1(k, l)<k^{3/2+\varepsilon}.$$

Analogous results are given for rth-power-free integers.

H. N. Shapiro (New York, N.Y.).

Whiteman, Albert Leon. The cyclotomic numbers of order sixteen. Trans. Amer. Math. Soc. **86** (1957), 401–413.

Let the prime $p=ef+1$, and let $(i, j)_e$ be the number of z for which $z\equiv g^{es+i}$ and $z+1\equiv g^{et+j} \bmod p$, with g primitive, and $1\leqq z\leqq p-2$. If $p=x^2+4y^2=a^2+2b^2$, and the signs of x and a are determined by $x\equiv a\equiv 1 \bmod 4$, while the signs of b and y are determined by the residue character of g

[see Dickson, Amer. J. Math. **57** (1935), 391–424], then $64(i, j)_8$ is a linear combination of p, x, y, a, b for each i, j. E. Lehmer proved by a calculation on the SWAC [Canad. J. Math. **5** (1953), 425–432; MR **15**, 10] that $256(i, j)_{16}$ is not a linear combination of the same variables for some i, j.

The author shows that if 2 is a biquadratic residue of $p=16f+1$ and $g^m \equiv 2 \bmod p$, then six relations exist of the type $256(0, 0)_{16}=p-47-18x$, for f even and $m \equiv 4 \bmod 8$. In general relations hold of the more difficult type $256(0, 0)_{16}=p-47-18x-48a+96d+48c$, for f even and $m \equiv 0 \bmod 8$. Here d, c are combinations of sums of the type

$$B(i, v)=\sum_{h=0}^{e-1}(h, i-vh).$$

From these results the author proves that the set of 16th power residues (augmented by zero or not) form no difference set if 2 is an octic residue of p, and rederives the Cunningham-Aigner 16th power residue character of 2 as in his earlier paper [Canad. J. Math. **6** (1954), 364–373; MR **16**, 14]. *H. Cohn* (Tucson, Ariz.).

Davenport, H. Indefinite quadratic forms in many variables. II. Proc. London Math. Soc. (3) **8**(1958), 109–126.

[For part I see Mathematika **3** (1956), 81–101; MR **19**, 19]. The author proves the following improvement of a previous result: Let $Q(x_1, \cdots, x_n)$ be an indefinite quadratic form with real coefficients. Suppose that when Q is written as a sum of squares of real linear forms with positive and negative signs, there are r positive signs and $n-r$ negative signs. Then the inequality $|Q(x_1, \cdots, x_n)|<\varepsilon$ is soluble in integers $x_1, \cdots, x_n$, not all 0, for any $\varepsilon>0$, provided that either $r \geqq 16$ and $n-r \geqq 16$ or $13 \leqq r \leqq 15$ and $n>(9r-20)/(r-12)$.

The improvement is based partially on Minkowski's theory of successive minima in the geometry of numbers. In a note added in August, 1957, the author calls attention to a method of B. J. Birch which is more effective for small values of r and which will appear in a joint paper. *B. W. Jones* (Boulder, Colo.).

Pall, G.; and Taussky, O. Application of quaternions to the representations of a binary quadratic form as a sum of four squares. Proc. Roy. Irish Acad. Sect. A. **58** (1957), 23–28.

Let $\varphi_1=e\varphi_l$, $\varphi_l=ax^2+2t_0xy+by^2$, where e, a, t_0, b are integers and l is the g.c.d. of a, t_0, and b. Let $r_s(n)$ denote the number of representations of the integer n as a sum of s squares of integers, and let $r_s(\varphi)$ denote the number of representations of φ as the sum of s squares of linear forms with integral coefficients. The authors derive the formula

$$r_4(\varphi)=\sum r_4(kh)r_3(e^2d;h),$$

where k is the g.c.d. of the coefficients of φ_l, h ranges over the positive divisors of s and $r_3(e^2d;h)$ denotes the number of solutions of

$$e^2d=t_1^2+t_2^2+t_3^2,\ (t_1, t_2, t_3, e)=h.$$

They also show that the function $f(e)=r_4(\varphi)/r_4(\varphi_l)$ is multiplicative in the restricted sense and from this fact derive a formula for $f(e)$ in terms of the prime power factors of e. *B. W. Jones* (Boulder, Colo.).

Rieger, G. J. Bemerkungen zu einem zahlentheoretischen Satz von Shapiro. Arch. Math. **8** (1957), 251–254.

An elementary derivation is given for the relation

$$\sum_{\substack{n \leqq x \\ (n,k)=1}} \frac{\mu(n)}{n} \log^r \frac{x}{n}=\frac{rk}{\varphi(k)} \log^{r-1} x+\sum_{m=1}^{r-2} b_m^{(r)} \log^m x+O(1),$$

where μ denotes the Möbius function, $r \geqq 2$ is an integer, and the $b_m^{(r)}$ are constants depending on m, r, and k. This generalizes a result of the reviewer [Ann. of Math. (2) **51** (1950), 485–497; MR **11**, 419] which corresponds to the case $k=1$ of the above. *H. N. Shapiro.*

Kanold, Hans-Joachim. Über das harmonische Mittel der Teiler einer natürlichen Zahl. II. Math. Ann. **134** (1958), 225–231.

Let $\tau(n)$ and $\sigma(n)$ be the number and sum, repectively, of divisors of an integer n. The author examines those n for which $n \cdot \tau(n)$ and $\sigma(n)$ are relatively prime. If $U(x)$ is the number of such $n \leqq x$, then it is shown that there exist positive constants c_1 and c_2 such that

$$c_1<U(x)x^{-\frac{1}{2}}(\log x)^{\frac{1}{2}}<c_2.$$

O. Ore (New Haven, Conn.).

Leopoldt, Heinrich-Wolfgang. Eine Verallgemeinerung der Bernoullischen Zahlen. Abh. Math. Sem. Univ. Hamburg **22** (1958), 131–140.

Let χ be a primitive character (mod f), where f is an arbitrary positive integer. The author defines B_χ^n, $B_\chi^n(x)$ by means of

$$\sum_{r=0}^{\infty} \chi(r) \frac{te^{(r+x)t}}{e^{ft}-1}=\sum_{n=0}^{\infty} B_\chi^n(x)\frac{t^n}{n!},\ B_\chi^n=B_\chi^n(0).$$

Then for $f=1$, B_χ^n, $B_\chi^n(x)$ reduces to the ordinary Bernoulli number and polynomial, respectively. The first section of the paper contains basic properties of the generalized numbers. In the second section the generalized numbers are expressed as singular values of Dirichlet L-series; this implies the non-vanishing of those numbers that are not obviously zero. The last section of the paper is devoted to an analog of the Staudt-Clausen theorem for the number B_χ^n. *L. Carlitz.*

Volkmann, Bodo. Über Hausdorffsche Dimensionen von Mengen die durch Zifferneigenschaften charakterisiert sind. V. Math. Z. **65** (1956), 389–413.

Given an integer $g \geqq 2$, every real number ρ, $0<\rho \leqq 1$, possesses a unique g-adic representation $\rho=\sum_{j=1}^{\infty} e_j g^{-j}$ where $\{e_j\}$ is an integer sequence containing infinitely many non-zero members and such that $0 \leqq e_j<g$; let $\{e_j\}$ be called the digital sequence of ρ. Let each of $F_1, F_2, \cdots, F_m$ be a finite, ordered set of non-negative integers less than g. The author computes the Hausdorff dimension α of the set of those numbers ρ, $0<\rho \leqq 1$, whose digital sequences do not contain any of $F_1, \cdots, F_m$ as a subset. He remarks that his proof of the special case $m=1$ in paper III of the series [Math. Z. **59** (1953), 259–270; MR **15**, 513] contains a fault which is now obviated by the methods of the present paper; these enable him also to find upper and lower bounds for the corresponding α-dimensional Hausdorff measure. The formula for α is given in a number of alternative forms, some referring to other special cases. *H. Halberstam* (London).

See also: Analytic Theory of Numbers: Venugopal Rao; Delange.

Analytic Theory of Numbers

Mitsui, Takayoshi. Generalized prime number theorem. Jap. J. Math. **26** (1956), 1–42.

The author proves the following theorem. Take an ideal number $\hat{p}$ [cf. Hecke, Math. Z. **6** (1920) 11–51] and

an ideal $\mathfrak{a}$ of an algebraic number field K of degree n. As usual, set $n=r_1+2r_2$, $r=r_1+r_2-1$. Let $Y_1, \cdots, Y_n$ be positive numbers such that $Y_p=Y_{p+r_2}$ $(p\geqq r_1+1)$ and $Y_i\leqq Y_j^a$ for a fixed positive number a. Further assume $(\hat{\rho}, \mathfrak{a})=1$ and $N\mathfrak{a}\leqq\log^A(y_1\cdots y_n)$ where A is a fixed positive number A.

Now take positive numbers $\vartheta_{r_1+1}, \cdots, \vartheta_{r_1+r_2}$ such that $0<\vartheta_p\leqq 1$ $(p=r_1+1, \cdots, r_1+r_2)$ and denote by $\pi(\mathfrak{a}, \hat{\rho}, y_q, \vartheta_p)=\pi(\mathfrak{a}, \hat{\rho}, y_1, \cdots, y_{r+1}, \vartheta_{r_1+1}, \cdots, \vartheta_{r_1+r_2})$ the number of the prime ideal numbers $\hat{\tilde{\omega}}$ satisfying: $\hat{\tilde{\omega}}\equiv\hat{\rho}$ (mod $\mathfrak{a}$), $|\hat{\tilde{\omega}}^{(q)}|\leqq y_q$ $(q=1, 2, \cdots, r+1)$, $0\leqq\arg\hat{\tilde{\omega}}^{(p)}<2\pi\vartheta_p$ $(p=r_1+1, \cdots, r_1+r_2)$.

Then we have

$$\pi(\mathfrak{a}, \hat{\rho}; y_q, \vartheta_p)=\frac{\vartheta_{r_1+1}\cdots\vartheta_{r_1+r_2}w}{2^{r_1}hq(\mathfrak{a})R}M + O(y_1\cdots y_n e^{-c\sqrt{(\log(y_1\cdots y_n))}}),$$

where

$$M=\int_2^{y_1^{e_1}}dt_1\cdots\int_2^{y_r^{e_r}}dt_r\int_2^{y_{r+1}^{e_{r+1}}}\frac{dt_{r+1}}{\log(t_1\cdots t_{r+1})},$$

$e_i=1$ $(i\leqq r_1)$, $e_i=2$ $(i\geqq r_1+1)$, h is the number of ideal classes, R is the regulator, w is the number of roots of unity in K, $c>0$, and the constants in the error term are dependent only on K, A and a but not on $\mathfrak{a}$, $\hat{\rho}$, $\vartheta_{r_1+1}, \cdots, \vartheta_{r_1+r_2}$.

Special cases of Mitsui's theorem are theorems of Rademacher [Math. Ann. **111** (1935), 207–228; Acta Arith. **1** (1935), 67–77]. *S. Chowla.*

Schoenfeld, Lowell. The order of the zeta function near the line $\sigma=1$. Duke Math. J. **24** (1957), 601–609.

Let $\zeta(s, w)$ stand for the Hurwitz zeta-function, and denote by $\mu(\sigma, w)$ the order of $\zeta(s, w)$, i.e., the greatest lower bound of the numbers μ_1, such that $|\zeta(s, w)|\leqq\beta|t^{\mu_1}$, where $\beta=\beta(\sigma)$. The following estimates [van der Corput and Koksma, Ann. Fac. Sci. Univ. Toulouse (3) **22** (1930), 1–39] are classical: If $Q=2^q$ and $\sigma=1-q/(Q-2)$, then, for integral $q\geqq 3$, $\mu(\sigma, w)\leqq 1/(Q-2)$; for $\frac{1}{2}\leqq\sigma\leqq 1$,

$$(*)\qquad \mu(\sigma, w)\leqq\frac{(1-\sigma)\log 2}{\log(1-\sigma)^{-1}}.$$

The author uses an estimate due to Vinogradov [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 109–130; MR **13**, 328; **16**, 1089] of $|\sum_{P<n<P'}(n+w)^{-s}|$, $1\leqq P\leqq P'<2P$, in conjunction with classical theorems on $\zeta(s, w)$, and proves: There exists a function $C(\omega)$, defined for every $\omega>\log 120\sqrt{2}$ and such that if $C(\omega)<\sigma<1$, then

$$(**)\quad \mu(\sigma, w)<\frac{\sqrt{2}}{3}(1-\sigma)^{3/2}\log^{\frac{1}{2}}(1-\sigma)^{-1} \times\left\{1-\frac{1}{2}\frac{\log\log(1-\sigma)^{-1}}{\log(1-\sigma)^{-1}}+\frac{\omega}{\log(1-\sigma)^{-1}}\right\}.$$

For $\omega=6$, in particular, the expression in braces may be replaced by 1. Clearly, (**) represents an improvementover (*), for values of σ close to $\sigma=1$. Assuming a certain generalization (to the best of the reviewer's knowledge, so far unproven) of Vinogradov's result, an estimate of the form $\mu(\sigma, w)\leqq C(a)(1-\sigma)^{1+1/a}$ follows, which is still better than (**), provided that one may use it with some $a\leqq 2$. Under certain conditions, the order of the Dirichlet series $L(s, \chi)$ is shown to admit the same bound as $\mu(\sigma, w)$. Finally, a simple proof is given of the fact that, for $1-\sigma$ sufficiently small, (*) may be improved by multiplying the second member by

$$\left\{1-\frac{\log\log(1-\sigma)^{-1}}{\log(1-\sigma)^{-1}}+\frac{\omega}{\log(1-\sigma)^{-1}}\right\}$$

for any $\omega>(\log 4)/e$. *E. Grosswald.*

Richert, Hans-Egon. Über Dirichletreihen mit Funktionalgleichung. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 73–124.

The author estimates partial sums $S_0(x)=\sum_{1\leqq n\leqq x}f(n)$, and more generally, $S_\kappa(x)=\sum_{1\leqq n\leqq x}f(n)(x-n)^\kappa$, where the $f(n)$ are coefficients of a Dirichlet series $Z(s)=\sum_1^\infty f(n)n^{-s}$. It is assumed that there is a second Dirichlet series $Z_1(s)$ which is related to $Z(s)$ by an equation of the type

$$Z(s)=C^{as+b}Z_1(\gamma-s)\prod_{j=1}^{M}\frac{\Gamma(\beta_j-b_js)}{\Gamma(\delta_j+d_js)}.$$

In the case $\kappa=0$, the problem was treated by Landau [Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl. **1912**, 687–771]. The present author removes some of Landau's supplementary conditions, and also considers $\kappa>0$, but his main achievement is that one of the major error terms is replaced by an exponential sum of the form

$$Cx^\eta\sum_{1\leqq n\leqq N}f_1(n)n^c\cos(\lambda(xn)^\rho+\Phi),$$

where C, η, c, λ, ρ, Φ are certain constants, and the $f_1(n)$ are the coefficients of $Z_1(s)$.

In many special cases non-trivial estimates for this exponential sum are available. The author uses estimates obtained in a previous paper [Math. Z. **58** (1953), 204–218; MR **15**, 11] in connection with the Dirichlet divisor problem. In this way he obtains the error term $O(x^{15/46+\epsilon})$ in the following problems: (i) The number of products $n_1n_2\leqq x$, where n_1 and n_2 are restricted to $n_1\equiv l_1 \pmod{k_1}$, $n_2\equiv l_2 \pmod{k_2}$; (ii) the number of ideals with norm $\leqq x$ in a quadratic number field; (iii) the coefficient problem for the product of two Dirichlet L-series; (iv) the coefficient problem for the Eisenstein series $G_1(\tau; a_1, a_2, N)$.

Furthermore, the author applies his results to questions about the abscissa of Riesz summability $(R, \log n, \kappa)$ of $Z(s)$. In particular he considers the Dirichlet series arising from modular cusp forms. *N. G. de Bruijn.*

Luthra, S. M. On the average number of summands in partition of n. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 483–498.

Let $p_k(n)$ denote the number of partitions of n into precisely k parts, $p(n)$ the total number of partitions of n. By the average number of summands A in a partition of n is meant $A=p(n)^{-1}\sum_{k=1}^n kp_k(n)$. By means of the identity $p(n)A=\sum_{k=1}^n d(k)p(n-k)$ ($d(k)$ the number of divisors of k), the asymptotic formula for $p(n)$, and the asymptotic formula of Szekeres for the value of k which maximizes $p_k(n)$, it is shown that

$$A=(CN^{\frac{1}{2}}+\tfrac{1}{2}C^2)(\log(CN^{\frac{1}{2}})+\gamma)+\frac{1+C^2}{4}+O(N^{-\frac{1}{2}}\log N),$$

where $C=6^{\frac{1}{2}}/\pi$, $N=n-1/24$, and γ is Euler's constant.

For the average number of summands B in a partition of n into distinct parts (which is similarly defined) it is shown that

$$B=2^{\frac{1}{2}}\log 2CM^{\frac{1}{2}}+\tfrac{1}{2}C^2\log 2-\tfrac{1}{4}+O(M^{-\frac{1}{2}}),$$

where $M=n+1/24$.

In addition asymptotic formulae for certain higher moments are worked out and tables of A, B and these moments are given for $1\leqq n\leqq 100$.

The paper is marred by several minor typographical errors and a disrespect for asymptotic relationships. In several places such relationships are replaced by equalities and in one instance such an equality is differentiated and the result made an asymptotic formula again. Nevertheless the results quoted seem to be correct, at least for the dominant term. *M. Newman* (Washington, D.C.).

Rademacher, Hans. On the Selberg formula for $A_k(n)$. J. Indian Math. Soc. (N.S.) **21** (1957), 41–55 (1958).

The arithmetical constants $A_k(n)$ in Rademacher's convergent series for the partition function are defined by

$$A_k(n)=\sum_{\substack{h \bmod k\\(h,k)=1}} \omega_{hk}\exp(-2\pi ihn/k),$$

where ω_{hk} are certain complicated $24k$th roots of unity arising from the theory of Dedekind's modular function $\eta(\tau)$. Selberg has discovered and the reviewer [Pacific J. Math. **6** (1956), 159–176; MR **18**, 195] has discussed the following much simpler form for $A_k(n)$:

$$3^{\frac{1}{2}}k^{-\frac{1}{2}}A_k(n)=\sum(-1)^j\cos(\pi(6j+1)/6k),$$

the sum extending over all j mod k for which

$$(3j^2+j)/2\equiv -n \pmod k.$$

The present author derives the Selberg formula directly from the transformation formula

$$\eta\Big(\frac{h+iz}{k}\Big)=\frac{\varepsilon}{z^{\frac{1}{2}}}\eta\Big(\frac{h'+iz^{-1}}{k}\Big),$$

where $hh'+1\equiv 0 \pmod k$ and $\varepsilon=\varepsilon(h,k)$ is a root of unity. The method of proof is elegant and does not require any evaluation of Gaussian sums. Lehmer's theorems [Trans. Amer. Math. Soc. **43** (1938), 271–295] for the evaluation and factorization of $A_k(n)$ are also deduced in a form which make the computations more transparent. In particular the author's improved version of the main multiplication theorem avoids any case distinctions with respect to the occurrence of prime factors 2 in k.

A. L. Whiteman (Los Angeles, Calif.).

Venugopal Rao, V. The lattice point problem for indefinite quadratic forms with rational coefficients. J. Indian Math. Soc. (N.S.) **21** (1957), 1–40 (1958).

Let $\mathfrak{x}$ be an m-dimensional, integral column vector, $\mathfrak{S}$ a real, rational, symmetric, positive definite, $m\times m$ matrix, $\mathfrak{a}$ and $\mathfrak{h}$ two fixed, real, m-dimensional, rational column vectors, denote the transpose by a dash, let $\mathfrak{S}[\mathfrak{x}]=\mathfrak{x}'\mathfrak{S}\mathfrak{x}$, and set

$$A(\mathfrak{S},\mathfrak{a},\mathfrak{h},t)=\sum_{\mathfrak{S}[\mathfrak{x}+\mathfrak{a}]=t} e^{2\pi i\mathfrak{x}'\mathfrak{h}}.$$

For $\mathfrak{a}=\mathfrak{h}=0$, the second member becomes the number of representations of the integer t by the quadratic form $\mathfrak{S}[\mathfrak{x}]$; in particular, if $\mathfrak{S}$ reduces to the unit matrix, this becomes the number of representations of t by a sum of m squares. If $\mathfrak{h}\neq 0$, each representation is weighted by the factor $e^{2\pi i\mathfrak{x}'\mathfrak{h}}$ and the expression is further generalized by taking $\mathfrak{a}\neq 0$. In the case of indefinite forms $\mathfrak{S}[\mathfrak{x}]$, instead of the (no longer finite) number of representations of t by $\mathfrak{S}[\mathfrak{x}]$, one has to consider Siegel's [Ann. of Math. (2) **37** (1936), 230–263] measure of representation. The author generalizes this concept to the present situation with $\mathfrak{h}\neq 0$ (the cases $m=3$, $-t|\mathfrak{S}|$ a square and $m=4$, $t=0$, $|\mathfrak{S}|$ a square excluded; also $m=2$, $-|\mathfrak{S}|$ a square). If $\mu(\mathfrak{S},\mathfrak{a},\mathfrak{h},t)$ is this (properly normalized) generalized measure of representation, one may define the zeta function $\zeta(\mathfrak{S},\mathfrak{a},\mathfrak{h},s)=\sum_{t>0}(\mathfrak{S},\mathfrak{a},\mathfrak{h},t)/t^s$. Let λ be a parameter, $0<\lambda<1$, $\mathfrak{X}$ a real matrix of rank n, $\mathfrak{S}[\mathfrak{X}]=\mathfrak{T}>0$, and set $\mathfrak{R}=\mathfrak{T}^{-1}[\mathfrak{X}'\mathfrak{S}]$, $\mathfrak{H}=\lambda\mathfrak{S}-\mathfrak{R}$. Furthermore, define the theta series $\vartheta(\xi,\eta,\lambda,\mathfrak{X},\mathfrak{S},\mathfrak{a},\mathfrak{h})=\sum\exp\{\pi i(\xi\mathfrak{S}-i\gamma\eta\mathfrak{H})[\mathfrak{x}+\mathfrak{a}]+2\pi i\mathfrak{x}'\mathfrak{h}\}$ where $\gamma=(\lambda(l-\lambda))^{-\frac{1}{2}}$ and the sum is taken over integral values of $\mathfrak{x}$. Integrating ϑ over a fundamental domain $F^*(\mathfrak{S})$, using an appropriate measure dv (both too complicated for exact definition here), one obtains

$$g(\xi,\eta,\lambda,\mathfrak{S},\mathfrak{a},\mathfrak{h})=\int_{F^*}(\mathfrak{S})\,\theta(\xi,\eta,\lambda,\mathfrak{X},\mathfrak{S},\mathfrak{a},\mathfrak{h})dv.$$

Using results of Siegel [Abh. Math. Sem. Hansischen Univ. **13** (1940), 209–239; Math. Ann. **124** (1951), 17–54; MR **2**, 148; **16**, 800], the author obtains the Fourier expansion of g. From this it follows that g is not an analytic function of $\xi+i\eta$, but satisfies a certain second order elliptic partial differential equation. The Mellin transform of $g(0,\eta,\lambda,\mathfrak{S},\mathfrak{a},\mathfrak{h})$ with respect to η is shown to be precisely $\zeta(\mathfrak{S},\mathfrak{a},\mathfrak{h},s)$, and use of the transformation formula of the theta function leads to the functional equation of this zeta-function. These results generalize those of Maass (S.-B. Heidelberger Akad. Wiss. Math. Nat. Klasse 1949, no. 1, 42 pp. (1949); MR **11**, 230] and Siegel [Math. Z. **43** (1938), 682–708]. ζ is in general meromorphic, but is actually an entire function of s if $\mathfrak{h}$ is not integral and either $|\mathfrak{S}|>0$, or $\mathfrak{S}^{-1}[\mathfrak{x}+\mathfrak{h}]$ is not a zero form. In all cases $(s-1)(s-\frac{1}{2}m)\zeta(\mathfrak{S},\mathfrak{a},\mathfrak{h},s)$ is an entire function of finite genus. Finally, using classical theorems and results from the theory of Bessel functions (partly due to the author), the author determines analytic expressions for the Riesz sums $\sum_{0<\lambda_l\leq x}(\mathfrak{S},\mathfrak{a},\mathfrak{h},\lambda_l)(x-\lambda_l)^\alpha$ for $x>0$ and $\alpha>\frac{1}{2}(m-1)$; here $\{\lambda_l\}$ is the sequence of positive values of $\mathfrak{S}[\mathfrak{x}+\mathfrak{a}]$. The formulae are different, according to the sign of $|\mathfrak{S}|$. Furthermore, if $|\mathfrak{S}|>0$, the Riesz summability of $\sum_{l=1}^{\infty}\mu(\mathfrak{S}^{-1},\mathfrak{h},-\mathfrak{a},\mu_l)J_\mu(2\pi(\mu_l x)^{\frac{1}{2}})\mu_l^{-\alpha}$, $\mu>-1$, α real, $x\neq\mu_l$ follows from Theorem 1 of Bochner [Ann. of Math. (2) **53** (1951), 332–363; MR **13**, 920]. Previously mentioned Riesz sums may be considered a generalization of Hardy's representation of $\sum_{0<n\leq x}r(n)(x-n)^\alpha$ ($r(n)$=number of representations of n as sum of two squares, $\alpha\geqq 0$, $x\neq$integer), by a series involving Bessel functions [Quart. J. Math. Oxford Ser. **46** (1915), 263–283; Proc. London Math. Soc. (2) **15** (1916), 192–213]. *E. Grosswald* (Philadelphia, Pa.).

Delange, H.; and Halberstam, H. A note on additive functions. Pacific J. Math. **7** (1957), 1551–1556.

Let $f(n)$ be an additive arithmetic function, and set

$$A_n=\sum_{p<n}f(p)/p,\quad B_n=\sum_{p<n}f^2(p)/p.$$

One of the problems concerning the distribution of values of additive arithmetic functions is the determination of conditions under which the distribution of

$$\frac{f(m)-A_n}{B_n^{\frac{1}{2}}}$$

is asymptotically Gaussian. This was proved first by Erdös and Kac [Amer. J. Math. **62** (1939), 738–742; MR **2**, 42] under the hypothesis

$$f(p)=O(1). \tag{1}$$

The reviewer later derived this [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 426–430; MR **18**, 113] under the weaker assumption

$$\lim_{n\to\infty}B_n^{-1}\sum_{\substack{p<n\\|f(p)|>\varepsilon B_n^{1/2}}}\frac{f^2(p)}{p}=0,\text{ for every }\varepsilon>0. \tag{2}$$

Delange [Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 130–144; MR **17**, 946] and Halberstam [J. London Math. Soc. **30** (1955), 43–53; MR **16**, 569] (independently), by means of moment calculations, had also derived the result under the assumption that

$$f(p)=o(B_p^k). \tag{3}$$

More precisely, they derived that for each fixed integer $k\geqq 1$,

$$\lim_{n\to\infty}\frac{\sum_{m=1}^{n}(f(m)-A_n)^k}{nB_n^{k/2}}=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}\omega^k e^{-\frac{1}{2}\omega^2}d\omega. \tag{4}$$

In this paper, they show that (4) may be derived from (2) together with

$$f(p)=O(B_p^k). \tag{5}$$

It is also shown that either (2) or (5) alone does not imply (4). *H. N. Shapiro* (New York, N.Y.).

Delange, Hubert. Sur certaines fonctions arithmétiques. C. R. Acad. Sci. Paris **245** (1957), 611–614.

Using some of his Tauberian theorems [Ann. Sci. Ecole Norm. Sup. (3) **73** (1956), 15–74; Colloque sur la Théorie des Nombres, Bruxelles, 1955, Thone, Liège, 1956, pp. 147–161; MR **18**, 720; **19**, 17] the author studies the distribution properties of the functions ω_E and Ω_E, which are respectively defined by a set E of prime numbers as the number of distinct prime factors and the total number of primes from E in the positive integer n. It is assumed that E is regular and of density α, i.e., there exists a non-negative number $\alpha\leqq 1$ such that, for $\Re(s)\geqq 1$, $\sum_{p\in E}P^{-s}=\alpha\log(s-1)^{-1}+r(s)$, where $r(s)$ is a holomorphic function for $\Re(s)\geqq 1$. The set E here is of the form: $E=E_1\cup E_2-E_3$, where (1) E_1 is the set, supposed infinite, of all primes belonging to a given arithmetical progression, or to the union of two or more given arithmetical progressions, (2) $E_2\cap E_1=\emptyset$ and $E_3\subset E_1$, and (3) the set E_j ($j=2, 3$) is empty or finite or the series $\sum_{p\in E_j}P^{-\sigma}$ is convergent for $\sigma<1$. Denote by (Φ) the family formed by all the sets of this form. Corresponding to the sets of the family (Φ) we designate by $(\mathfrak{F})$ the family formed by all functions ω_E and Ω_E. In order to study the distribution of values of a function of the family $(\mathfrak{F})$ the method is to evaluate the series $P_x(z)=\sum_{n\leqq x,\,n\in A}z^{f(n)}$, where z is a complex variable. Several asymptotic formulae are obtained by imposing various conditions on f and the set A.

S. Ikehara (Tokyo).

Delange, Hubert. Sur certaines fonctions arithmétiques. C.R. Acad. Sci. Paris **245** (1957), 1197–1200.

This paper treats two functions simultaneously of the family $(\mathfrak{F})$ (terminology same as in the preceding review). Let f be one of the functions ω_{E_1} and Ω_{E_1}, and g one of the functions ω_{E_2} and Ω_{E_2}, E_1 and E_2 being two disjoint sets of the family (Φ). Let us denote by α and β the respective densities of E_1 and E_2 with respect to the set of all prime numbers. If A is the set of all positive integers, or set of squarefree integers, the author derives, when $x\to\infty$, a formula of the form:

$$\sum_{n\leqq x,\ n\in A}u^{f(n)}v^{g(n)}=$$
$$G(u,v)x(\log x)^{\alpha u+\beta v-\alpha-\beta}+O(x(\log x)^{\alpha Ru+\beta Rv-\alpha-\beta-1})$$

valid uniformly for $|u|\leqq u_0$, $|v|\leqq v_0$ if u_0 and v_0 are positive numbers satisfying $u_0<U$, $v_0<V$, where U and V are appropriately defined, and $G(u, v)$ is holomorphic in u and v. From the above formula are deduced several asymptotic expressions for the number of positive integers $n\leqq x$ by giving conditions, among others, on both $f(n)$ and $g(n)$. *S. Ikehara* (Tokyo).

Knapowski, S. Über ein Problem der Gleichverteilungstheorie. Colloq. Math. **5** (1957), 8–10.

Let $a_1, a_2, \cdots$ be an increasing sequence of positive integers. The author studies the distribution of the sequence $a_1^{-1}, 2a_1^{-1}, 3a_1^{-1}, \cdots, (a_1-1)a_1^{-1}, a_2^{-1}, 2a_2^{-1}, \cdots, (a_2-1)a_2^{-1}, a_3^{-1}, \cdots$, abbreviated by $u_1, u_2, \cdots$. If $0\leqq\alpha<\beta\leqq 1$, and $A(n)$ is the number of u_i with $\alpha<u_i\leqq\beta$, $i<n$, then $A(n)=(\beta-\alpha)n+O(a_k)$, where a_k is the denominator involved in u_n. Unless $\alpha=\beta-1=0$, the symbol O cannot be replaced by o. In the proof of the latter statement the author assumes that $a_k/k\to\infty$, but it is easy to see that this assumption is superfluous.

N. G. de Bruijn (Amsterdam).

See also: General Theory of Numbers: Volkman. **Integral Transforms:** Wintner.

Theory of Algebraic Numbers

Degert, Günter. Über die Bestimmung der Grundeinheit gewisser reell-quadratischer Zahlkörper. Abh. Math. Sem. Univ. Hamburg **22** (1958), 92–97.

For real quadratic fields generated by a square-free $D=n^2+r$, $-n<r\leqq n$, when r divides $4n$, then $(4n^2+2r+4nD^{\frac{1}{2}})/2|r|$ is a unit. Thus the fundamental unit can be seen to have norm $+1$ (unless $r=1$ or 4), enabling the author to show in cases like $D=6^2-2=34$ that the fundamental unit has norm $+1$ (without, for instance, having a prime factor in D of the form $4m-1$).

H. Cohn (Tucson, Ariz.).

Rieger, G. J. Über die Anzahl der Teiler der Ideale in einem algebraischen Zahlkörper. Arch. Math. **8** (1957), 162–165.

Let K be a given algebraic number field of degree n over the rationals, $\mathfrak{f}$ a fixed integral ideal of this field, $\mathfrak{h}$ an ideal-class mod $\mathfrak{f}$, h the number of such ideal-classes; and denote by $d(\mathfrak{a})$ the number of divisors of the integral ideal $\mathfrak{a}$. The author derives that

$$\sum_{\substack{N\mathfrak{a}\leqq x\\ \mathfrak{a}\in\mathfrak{h}\bmod\mathfrak{f}}}d(\mathfrak{a})=\frac{\alpha^2}{h}\prod_{\mathfrak{p}\nmid\mathfrak{f}}\left(1-\frac{1}{N\mathfrak{p}}\right)^2 x\log x+\alpha_2 x+O(x^{1-1/(n+1)}),$$

where α is a constant depending on K, and α_2 a constant depending on K and $\mathfrak{f}$. *H. N. Shapiro.*

Whaples, G. Galois cohomology of additive polynomial and n-th power mappings of fields. Duke Math. J. **24** (1957), 143–150.

Let k be a discrete valued complete field whose residue class field $\bar{k}$ has no inseparable algebraic extension and has for each n at most one algebraic extension of degree n. The main result of the paper is the proof of the following

statement: If $\bar{k}^{\cdot 2}=\bar{k}^{\cdot}$, then: (A) the Reciprocity Law, i.e., Th. 3 of Whaples, same J. **19** (1952), 505–517 [MR **14**, 140], holds for all abelian extensions K/k if $\bar{k}$ has an extension of degree $[K:k]$; (B) (Existence Theorem) every analytic $\mathfrak{N}^{\cdot}$ of $k^{\cdot}$ is norm group of an abelian extension K/k if $\bar{k}$ has an extension of degree $[k^{\cdot}:\mathfrak{N}^{\cdot}]$. The same statements remain true for $\bar{k}^{\cdot 2}\neq\bar{k}^{\cdot}$ if $[K:k]$, $[\bar{k}^{\cdot}:\mathfrak{N}^{\cdot}]$ are restricted to be odd. The proof makes use of Galois cohomology of additive polynomial and nth power mappings of fields, which leads to interesting intermediary results. *C. Arf* (Istanbul).

Geometry of Numbers

See: General Theory of Numbers: Davenport.

ANALYSIS

★**Bass, J. Cours de mathématiques. Algèbre linéaire; intégrales simples et multiples; séries et intégrales de Fourier; fonctions analytiques; équations différentielles et aux dérivées partielles; éléments de calcul numérique; mécanique et graphique.** Préface de G. Darmois. Masson et Cie, Paris, 1956. xi+916 pp. 7800 francs.

This book is written for the physicist, the engineer, and the student of applied mathematics. The author has not included important abstract topics which do not have immediate applications, for example, general topology and the theory of measure. It results from this that some theorems which are necessary for applications are stated without proof, such as the theorem of Fubini and the Riesz-Fisher theorem. On the other hand, proofs whose methods are useful in applications, or which are models of elegant and incisive mathematical reasoning, have been included. Considering the author's care and skill in selecting material, the length of the book can be taken as a fair measure of the extent to which modern physics and engineering practices are involved in and depend on mathematics.

The book is written in eight parts, each with a bibliography and a substantial list of exercises. There are two appendices (on the calculus of variations and nomography), a list of special series, and tables of integrals.

Part I, in five chapters and 82 pages, under the general heading linear algebra, deals with vector spaces, matrices, Hermitian spaces, the algebra of tensors and the applications of tensors. It is indicative of the trend of the times that a book on applied mathematics, written for physicists and engineers, begins by laying a solid groundwork in linear algebra. There are examples of applications to differential equations, geometry and dynamics.

Parts II and III, Ch. VI to VIII inclusive, deal with the usual topics of real functions, sequences and series that are found in American and Canadian universities in courses labelled advanced calculus. Chs. VIII and IX, pp. 145–188, cover formal integration, including elliptic integrals, the representation of integrals by Bernoulli numbers and Bernoulli polynomials, and the approximation of definite integrals by such means as the trapezoidal rule, Simpson's rule, and the Euler-Maclaurin formula. Ch. X deals with the more elementary properties of series of numbers, and with generalized integrals, both absolutely and non-absolutely convergent. Part III, pp. 226–340, Chs. XI to XV inclusive, deals with series of functions, integrals containing a parameter, Fourier series and integrals. The usual results relevant to Fourier series are demonstrated, with some brief and elegant proofs, leading to a consideration of orthogonal functions, a statement of the Riesz-Fischer theorem, the mean square convergence theorem and Parseval's formula. Ch. XV gives the development of the Fourier integral, with applications. Part III ends with a list of 54 problems many of which lead to additional important results.

Coming to Part IV, Ch. XVI is a study of curves using vector notation, and Ch. XVII gives the usual discussion of curvilinear integrals and integration of total differentials; Ch. XVIII is unusual in that it deals at some length with the ideas of mechanical integration. There are photographs of the Amsler planimeter, integrator and integraph, and explanations of the theory involved in their use. There are also photographs of the Coradi integraph and harmonic analyzer with explanations.

Chs. XIX–XXIII deal with double and multiple integrals, change of variable in double integrals, surface integrals and the integral formulas of vector analysis. Ch. XXV gives the theory and some applications of the Gamma function, which leads to the development of the integrals of Wallis and Stirling's formula. Ch. XXVI is an introduction to and applications of the Laplace transform.

Part VI, pp. 564–672, is given over to analytic functions. There are all the usual developments, including Taylor's and Laurent's series, analytic extension, the theory of residues, ending with a proof of the complex inversion theorem for Laplace transforms.

The remaining two parts of the book deal with differential systems and equations, and are slanted towards applications. There is a chapter on Bessel's functions and a chapter on numerical solution of differential equations. The last two chapters are given over to linear partial differential equations of the second order. There is a general discussion followed by a detailed study of the equations of a vibrating string and vibrating membrane, plane, sinusoidal and spherical waves, and the equations of heat flow. The final chapter deals with Laplace's equation in two and three variables, harmonic functions and Newtonian potentials. There are solutions of Laplace's equation in three variables by means of cylindrical coordinates, and by means of spherical coordinates, which introduces Legendre polynomials and the spherical functions of Laplace. It is shown that the volume potential satisfies Poisson's equation.

In conclusion, the reviewer does not hesitate to assert that the author has succeeded remarkably well in his avowed purpose of showing clearly what principles of mathematics are involved, of avoiding long and tiresome arguments, yet, with few exceptions, giving complete proofs. Not only is the style sustained and inspiring for all of the 900 pages, but the format is attractive. The outcome is something which every student and teacher of mathematics, and every physicist and engineer who is using mathematics, should have at hand. *R. L. Jeffery.*

Functions of Real Variables

★ **Hobson, E. W. The theory of functions of a real variable and the theory of Fourier's series. Vol. I.** Dover Publications, Inc., New York, N.Y., 1958. xv+736 pp. $3.00.

★ **Hobson, E. W. The theory of functions of a real variable and the theory of Fourier's series. Vol. II.** Dover Publications, Inc., New York, N.Y., 1958. x+780 pp. $3.00.

The first edition of this well-known work was published in 1907. The present reprint, first published in 1957, is an unabridged and unaltered republication of the third edition of 1927.

Block, H. D. A class of inequalities. Proc. Amer. Math. Soc. **8** (1957), 844–851.

Block, H. D. Discrete analogues of certain integral inequalities. Proc. Amer. Math. Soc. **8** (1957), 852–859.

Though the author does not formulate all of his results explicitly, these papers deal with the following theorem, its finite analogues, and some of their special cases. Let $y(x)$, $w(x;t)$, $g(x)$ be continuous functions of x in $[a,b]$; let $f(x)$ have continuous, and $w(x;t)$ and $y(x)$ have sectionally continuous derivatives there (derivation always with respect to x); suppose $(fw')'-gw=0$, for $a\leqq x<t$ and $t<x\leqq b$; $\lim_{\varepsilon\to 0}[w'(t-\varepsilon;t)-w'(t+\varepsilon;t)]f(t)=1$; and $f(a)w'(a)y(a)=f(b)w'(b)y(b)$; then $y(t)\leqq$

$$[\int_a^b(|f^\alpha w'|^p+|g^\gamma w|^p)^{\mu/p}dx]^{1/\mu}[\int_a^b(|f^\beta y'|^q+|g^\delta y|^q)^{\nu/q}dx]^{1/\nu},$$

where $\alpha+\beta=\gamma+\delta=1/p+1/q=1/\mu+1/\nu=1$; $p\geqq 1$; $\mu\geqq 1$.
J. Aczél (Debrecen).

Rajagopal, C. T. Some theorems on convergence in density. Publ. Math. Debrecen **5** (1957), 77–92.

In several papers, the reviewer has explored some aspects of a generalized asymptotic density and its applications [Amer. J. Math. **75** (1953), 335–346; Quart. J. Math. Oxford Ser. (2) **6** (1955), 128–131; MR **14**, 854; **17**, 253]. The investigations in the present paper constitute extensions of these, chiefly for the case of ordinary linear density. One says that $f(x)$ converges to L "in density" as $x\to\infty$ if it is true that $f(x)$ does in fact converge to L after a set of zero (linear) density has been removed from the domain of f. In the cited papers, a general theorem was proved which yielded as a corollary the following typical result: If $\liminf_{x\to\infty} f(x)\geqq L$ and $(C,1)$-$\lim_{x\to\infty} f(x)=L$, then $f(x)$ converges to L in density. The author of the present paper discusses extensions of this special result, and adds a number of interesting further applications in analysis. Among these, the following is typical. Let ψ be bounded, positive, with $\int_0^\infty \psi=1$, and set $F(y)=y\int_0^\infty \psi(yt)f(t)dt$. Then, if f is bounded and $(C,1)$-$\limsup_{x\to\infty} f(x)=\limsup_{x\to\infty} f(x)=L$ it follows that $\limsup_{y\downarrow 0} F(y)=L$. *R. C. Buck.*

Sandor, Stefan. Sur la convergence vers une fonction continue d'une suite de fonctions arbitraires. Rend. Mat. e Appl. (5) **16** (1957), 430–446.

Let $\{f_n\}$ be a sequence of real-valued functions on a real interval I, and let x^* be a point of I. The sequence $\{f_n\}$ is said to converge almost continuously at x^* if to every sequence $\{x_n\}$ of points of I converging to x there corresponds an increasing sequence $\{\nu_k\}$ of natural numbers such that for every sequence $\{n_k\}$ of natural numbers for which each $n_k\geqq\nu_k$, the limit $\lim_{k\to\infty} f_{n_k}(x_k)$ exists independently of the particular such sequence $\{n_k\}$. Among other things it is shown that if $\{f_n\}$ converges pointwise on I to a function f, then f is continuous at x^* if and only if $\{f_n\}$ converges almost-continuously at x^*. A notion of equi-almost-continuity of $\{f_n\}$ at x^* is introduced, in terms of which an Arzèla-type theorem concerning almost-continuous convergence is formulated and proved. Also, for the limit f of a pointwise convergent sequence $\{f_n\}$, measurability and possession of the property of Baire are each characterized in terms of almost-continuous convergence of $\{f_n\}$. *T. A. Botts.*

Rodnyanskiĭ, A. M. On differentiable maps. Aviacion. Inst. Sergo Ordžonikidze. Trudy Inst. no. **61** (1956), 58–67. (Russian)

Detailed proofs of results announced earlier [Dokl. Akad. Nauk SSSR (N.S.) **72** (1950), 15–17; MR **11**, 718]. *E. E. Floyd* (Charlottesville, Va.).

Lorenzen, Paul. Differentialformen und mehrdimensionale Integrale. Math.-Phys. Semesterber. **5** (1957), 200–213.

This is a cogent reexamination of certain aspects of elementary calculus, done in the spirit of some of the more recent texts, and with particular reference to the notion of the differential of a function or a transformation. The plane is a space of points P; a coordinate system is a mapping of the plane into E^2, the real two-dimensional vector space, given by two functions x_1 and x_2, so that the coordinates of P are the pair $(x_1(P), x_2(P))$. With a numerical-valued function g defined on E^2, one may associate a numerical function f defined on the plane by $f(P)=g(x_1(P), x_2(P))$; one may thus write $f=g(x_1, x_2)$ without ambiguity. One then understands $\partial f/\partial x_1$ to be the numerical-valued function defined on the plane (with the assigned coordinate system x_1, x_2) by

$$\frac{\partial f}{\partial x_1}(P)=g_1(x_1(P), x_2(P)),$$

where g_1 is the usual partial derivative with respect to the first position. The differential df then becomes a function defined on coordinate systems, whose values are pairs of point functions:

$$[df](\tilde{x}_1,\tilde{x}_2)=\Big(\frac{\partial f}{\partial\tilde{x}_1},\frac{\partial f}{\partial\tilde{x}_2}\Big)=$$

$$\frac{\partial f}{\partial x_1}[dx_1](\tilde{x}_1,\tilde{x}_2)+\frac{\partial f}{\partial x_2}[dx_2](\tilde{x}_1,\tilde{x}_2).$$

In one variable, one has recovered suitable meanings for the customary formulas $d\sin x=\cos x\,dx$, and $d\sin x/dx=\cos x$. The point of view has, of course, much in common with current practice in differential geometry, and with similar suggestions made by Menger, Duren, Monroe and others. *R. C. Buck* (Palo Alto, Calif.).

R. Salinas, Baltasar. Zeros of the functions of a class not quasi-analytic in R^n. Non-quasi-analytic continuation. Collect. Math. **9** (1957), 65–76. (Spanish)

The author defines quasianalytic classes of functions in n-space: the bounds m_n for $|f^{(n)}(x)|$ are replaced by simultaneous bounds m_n for all partial derivatives of total index n. He extends a number of theorems from the one-dimensional case to the n-dimensional case. Among these are: The familiar characterization of quasianalytic

classes by $\{m_n\}$. A non-quasianalytic class contains a non-negative element which is zero exactly on a given closed set. If two disjoint closed sets are given and one is compact, a non-quasianalytic class contains an element whose values lie between 0 and 1 and which takes the values 0 and 1 on two sets. A function analytic on a compact set has an extension in a given non-quasianalytic class which vanishes outside a sufficiently large sphere.

R. P. Boas, Jr. (Evanston, Ill.).

Measure, Integration

Ranga Rao, R. A note on finitely additive measures. Sankhyā **19** (1958), 27–28.

The supremum of any linearly ordered (bounded) family of countably additive non-negative finite measures on an algebra of sets is again countably additive. This variant of the Vitali-Hahn-Saks-Nikodým theorem implies a result of K. Yosida and the reviewer [Trans. Amer. Math. Soc. **72** (1952), 46–66; MR **13**, 543], Theorem 1.23.

E. Hewitt (Seattle, Wash.).

Dinculeanu, N. Remarques sur les mesures dans les espaces produits. Colloq. Math. **5** (1957), 51–54.

Soit J un ensemble d'indices et pour tout $j \in J$ soit X_j un espace et T_j une tribu ($=\sigma$-algèbre) de parties de X_j. Soit $E=\prod_{j\in J} X_j$ et pour toute partie finie $I \subset J$ soient $E^I=\prod_{j\in I} X_j$, pr_I la projection de E sur E^I et $T^I=\prod_{j\in I} T_j$. Soit T la réunion des tribus $\mathrm{pr}_I^{-1}(T^I)$ et soit $P(A)$ une probabilité définie sur T, dénombrablement additive sur chaque tribu $\mathrm{pr}_I^{-1}(T^I)$. Dans cet article l'auteur démontre deux théorèmes qui donnent des conditions suffisantes pour que $P(A)$ soit dénombrablement additive sur T et par suite prolongeable à la tribu engendrée par T. Les conditions introduites sont exprimées à l'aide des probabilités conditionnées $P(\mathrm{pr}_I(x)=z^I, \mathrm{pr}_t^{-1}(A_t))$. L'article se termine par quelque remarques sur les relations entre différentes conditions qui impliquent chacune l'additivité dénombrable de $P(A)$.

C. T. Ionescu Tulcea (New Haven, Conn.).

Goetz, A. Bemerkungen über Hausdorffsche Masse und Hausdorffsche Dimensionen in Lieschen Gruppen. Colloq. Math. **5** (1957), 55–65.

Let $\rho(x, y)$ be a left-invariant metric in an n-parameter Lie group G with unit element e. Put $2^a=\limsup(\rho(e, x^2)/\rho(e, x))$ and $2^b=\liminf(\rho(e, x^2)/\rho(e, x))$ as $x\to e$. It is shown that $0\leq a\leq 1$ and that the Hausdorff dimension h_ρ of G with respect to ρ satisfies $n/a\leq h_\rho$. In case $b>0$ it also satisfies $h_\rho\leq n/b$. For any $w>0$, the w-dimensional Hausdorff measure Λ_w corresponding to ρ is a left-invariant measure in G, though possibly trivial. When $w\neq h_\rho$, Λ_w is certainly not a Haar measure. When $w=h_\rho$, Λ_w is still not necessarily a Haar measure, even when $a=b=1$, as is shown, for example, by taking $\rho(x, y)=-|x-y|\log|x-y|$, $|x-y|\leq 1/e$, in the additive group of real numbers. But if $\rho(e, x^2)/\rho(e, x)$ is constant in a neighborhood of e, then $a=b$ and $\Lambda_{n/a}$ is a left-invariant Haar measure in G. In particular, Λ_n is a Haar measure for any left-invariant Riemannian metric ρ in the component of e.

J. C. Oxtoby (Bryn Mawr, Pa.).

Holladay, John C. On the existence of a mixing measure. Proc. Amer. Math. Soc. **8** (1957), 887–893.

The author considers a transformation f of [0, 1] into itself. He supposes that there exists a finite point set P in the interval, containing 0, 1 and at least one other point, and that (a) $f(P)\subset P$; (b) f maps each open component C of $[0, 1]-P$ homeomorphically; (c) the endpoints of $f(C)$ are in P; (d) for some integer $n>0$, $f^n(C)-P=[0, 1]-P$; (e) df/dx is uniformly continuous on C and there is an integer $p>0$ with the property that $|df^p/dx|$ on the inverse image under f^{p-1} of $[0, 1]-P$ has a strictly positive infimum. It is proved that, under these hypotheses, there is a unique F in L_1 such that, if μ is the measure with density F, $\mu[0, 1]=1$ and the transformation f is μ-measure preserving, $\mu(f^{-1}(E))=\mu(E)$ whenever E is Lebesgue measurable, and the transformation has the strong mixing property. Then the ergodic theorem in the form given by the reviewer [Duke Math. J. **6** (1940), 290–306; MR **1**, 344] can be applied to this not 1-1 measure preserving transformation.

J. L. Doob.

Kober, H. On the existence of the Burkill integral. Canad. J. Math. **10** (1958), 115–121.

Let R be an n-dimensional interval and f a real function of the n-dimensional interval $I\subset R$, $n\geq 1$. Consider the following statements: (a) f is "infinitesimally additive" on R, i.e., for every fixed $(n-1)$-dimensional interval i, the relations $I_1I_2=i$, $I=I_1+I_2\subset R$ and $|I|\to 0$ imply $f(I_1)+f(I_2)-f(I)\to 0$; (b) $|f|$ has a finite upper Burkill integral over R; (c) the lower Burkill integral of f over R is bounded above; (d) f is Burkill integrable over R. The main result of the paper is: If f increases by subdivision, then (d) is implied by the logical product of (a) and (b) and, for $n=1$, is equivalent to the logical product of (a) and (c).

H. M. Schaerf (Madison, Wis.).

Nakanishi, Shizu. L'intégrale de Denjoy et l'intégration au moyen des espaces rangés. II. Proc. Japan Acad. **33** (1957), 13–18.

Nakanishi, Shizu. L'intégrale de Denjoy et l'intégration au moyen des espaces rangés. III. Proc. Japan Acad. **33** (1957), 265–270.

Suite et fin de la démonstration du dernier théorème d'une note précédente [mêmes Proc. **32** (1956), 678–683; MR **19**, 256].

A. Appert (Angers).

Bauer, Heinz. Sur l'équivalence des théories de l'intégration selon N. Bourbaki et selon M. H. Stone. Bull. Soc. Math. France **85** (1957), 51–75.

The abstract theory of integration presented by M. H. Stone [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 336–342, 447–455, 483–490; **35** (1949), 50–58; MR **10**, 24, 107, 239, 360] includes the Bourbaki theory of integration on locally compact spaces [Eléments de mathématique, XIII, XXI, Hermann, Paris, 1952, 1956; MR **14**, 960; **18**, 881] as a special case. S. Kakutani [Ann. of Math. (2) **42** (1941), 523–537; MR **2**, 318] showed that the two theories are equivalent in the sense that an abstract Stone measure could be represented as a Radon measure on a suitably chosen compact space. This representation has, however, the following three properties which may be undesirable. 1) The locally compact space determined depends on the abstract measure μ as well as on the base space E and the vector space $\mathfrak{R}$ of functions on E which form the point of departure of the Stone theory. 2) If applied to the Radon measure μ on a locally compact space E, the locally compact space $\tilde{E}$ of the representation is not in general E and may have a more complex topological structure. 3) The method of representation does not apply to the problem of representing abstract Rie-

mann integration by Riemann integration on locally compact spaces.

The primary purpose of the present paper is to develop a representation theory in which the basic vector space $\mathscr{R}$ of the abstract theory has a more direct image in the space of continuous functions on the locally compact space, and which does not have the defects listed above. In order to do this the author finds it necessary to add the following hypothesis on the space $\mathscr{R}$ to those of the Stone theory: If $f \in \mathscr{R}$, then inf $(1, f) \in \mathscr{R}$ (1 denoting the function on E with values identically equal to 1). The subspace $\mathscr{R}^0$ of $\mathscr{R}$, consisting of the bounded functions g of $\mathscr{R}$ for which there is a function h of $\mathscr{R}_+$ such that $h(x)<1$ implies $g(x)=0$ for all $x \in E$, is introduced. Extensive use of this subspace is made to extend the results presented in Bourbaki [loc.cit.] on $L_F^p(E, \mathscr{R}, \mu)$ and $L_F^{p*}(E, \mathscr{R}, \mu)$ the spaces of functions with values in a Banach space F and with pth power μ-integrable by, respectively, the first and second integration procedure of Stone. The following representation theorem is then obtained. There exists a locally compact space E' and a positive linear mapping $\mathbf{T}$ of the Riesz space $\mathscr{J}(\mathscr{R})$ of relatively bounded linear forms on $\mathscr{R}$ into the Riesz space $\mathscr{M}(E')$ of Radon measures on E' with the following properties: For every abstract measure μ on $\mathscr{R}$, every real number $1 \leq p < +\infty$ and every Banach space F, the spaces $L_F^p(E, \mathscr{R}, \mu)$ and $L_F^{p*}(E', \mathscr{K}(E'), \mathbf{T}(\mu))$ are isomorphic as Banach spaces and in the case $F=\mathbf{R}$ the same isomorphism applies to their structure as Riesz spaces. The restriction of $\mathbf{T}$ to the set of abstract measures on $\mathscr{R}$ is one-to-one.

W. R. Transue (Gambier, Ohio).

Cesari, Lamberto. Fine-cyclic elements of surfaces of the type v. Riv. Mat. Univ. Parma 7 (1956), 149–185.

Let T be a single valued continuous mapping from a closed finitely connected Jordan region J in two-space into n-space. Denote by Γ the class of all the disjoint maximal continua of constancy for T in J. A continuum K in J is said to have property P_1 with respect to (T, J) if for every component v of $J-K$ the mapping T is constant on each component of the boundary of v in J. A continuum K in J is said to have property P_2 with respect to (T, J) if for every component v of $J-K$ the common part of v and its boundary is either empty or a subset of exactly one of the curves bounding J. A continuum K in J is said to have property P' with respect to (T, J) if every element of Γ which intersects K is a subset of K. A non-empty continuum K in J is defined to be a fine cyclic element of (T, J) if (1) K has both properties P_1 and P'; (2) T is not constant on K; (3) K is minimal with respect to properties (1) and (2). For each point w_0 in J let Ω denote the set of all points w in J for each of which there does not exist a finite subset of Γ whose union separates w_0 from w in J. It is shown that either Ω is the element of Γ containing w_0 or there are fine cyclic elements of (T, J) containing w_0 and Ω is the union of all such. It is proved that the class of fine cyclic elements of (T, J) is countable. Each fine cyclic element K_i of (T, J) is contained in a finitely connected closed Jordan region J_i contained in J and such that K_i has properties P_1, P_2, P' with respect to (T, J_i). The mapping T_i defined on J_i by the requirements that it agree with T on K_i, and map each component of J_i-K_i onto the point onto which T maps the boundary of that component relative to J_i, is termed the retraction of (T, J) with respect to K_i in J_i. It is shown that the Lebesgue area $L(T_i, J_i)$ of the surface represented by (T_i, J_i) is independent of the choice of the region J_i subject to the conditions described above. If $L(T, J)$ denotes the Lebesgue area of the surface represented by (T, J) it is proved that $L(T, J)=\sum L(T_i, J_i)$.

P. V. Reichelderfer (Columbus, Ohio).

Mickle, E. J.; and Neugebauer, C. J. Weak and strong cyclic additivity. Riv. Mat. Univ. Parma 7 (1956), 243–253.

Let $\mathscr{T}$ denote the class of all continuous mappings T from a Peano space P into a metric space P^*. An unrestricted factorization of a T in $\mathscr{T}$ is a collection $(\mathscr{M}, f, s)$ consisting of a Peano space $\mathscr{M}$, a continuous mapping f from P into $\mathscr{M}$, a continuous mapping s from $\mathscr{M}$ into P^*, such that $T=sf$. An unrestricted factorization $(\mathscr{M}, f, s)$ of a T in $\mathscr{T}$ is monotone light if f is a monotone mapping of P onto $\mathscr{M}$ and s is a light mapping of $\mathscr{M}$ into P^*. Denote by r_C the monotone retraction of $\mathscr{M}$ onto one of its proper cyclic elements C. Let Φ be a functional defined for every element T in $\mathscr{T}$ whose values are either non-negative real numbers or $+\infty$. The Φ is termed weakly additive if for every T in $\mathscr{T}$ and every monotone light factorization $(\mathscr{M}, f, s)$ of T it is true that $\Phi(T)=\sum \Phi(sr_C f)$, where the summation extends over all the proper cyclic elements C of $\mathscr{M}$. The Φ is termed strongly additive if for every T in $\mathscr{T}$ and every unrestricted factorization $(\mathscr{M}, f, s)$ of T it is true that $\Phi(T)=\sum \Phi(sr_C f)$. If P is unicoherent and Φ is weakly additive it is proved that Φ is strongly additive. An example is given to show that the assumption that P is unicoherent cannot be deleted. *P. V. Reichelderfer.*

Neugebauer, Christoph J. A cyclic additivity theorem of the Lebesgue area. II. Riv. Mat. Univ. Parma 7 (1956), 283–292.

A subset of two-space E_2 which is the union of a finite number of disjoint finitely connected polygonal regions is called a figure. Let A be a subset of E_2 which is either a figure or for which there exists a sequence of figures F_n each contained in the interior A^0 of A with the property that if K is any compact set in A^0 then K is also in F_n^0 for all but a finite number of the n. Given a continuous mapping T from A into three-space E_3, denote by $L(T, A)$ the Lebesgue area of the surface determined by restricting T to A^0. An unrestricted factorization of T is a collection $(\mathscr{M}, \mathscr{M}^*, f, s)$ consisting of a Peano space $\mathscr{M}$, a subset $\mathscr{M}^*$ of $\mathscr{M}$, a continuous mapping f from A into $\mathscr{M}^*$, a continuous mapping s from $\mathscr{M}^*$ into E_3, such that $T=sf$. If C is a proper cyclic element of $\mathscr{M}$ let r_C be the monotone retraction of $\mathscr{M}$ onto C, and denote by G_C the union of those components G of A for which $r_C f|G$ is contained in $\mathscr{M}^*$. Let $\mathscr{K}$ be the class of those proper cyclic elements C of $\mathscr{M}$ for which G_C is not empty. It is shown that $L(T, A)=\sum L(sr_C f, G_C)$, $C \in \mathscr{K}$. *P. V. Reichelderfer.*

Neugebauer, Christoph J. A further extension of a cyclic additivity theorem of a surface integral. III. Riv. Mat. Univ. Parma 7 (1956), 333–347.

Let T be a continuous transformation from an admissible set A in two-space [L. Cesari, Surface area, Princeton Univ. Press, Princeton, 1956, § 5.1; MR 17, 596] into three-space for which the surface represented by T has finite Lebesgue area. Assume that $F(p, q)$ is a function defined and continuous for every point p in a compact set X containing TA and for every vector triple q, which is positively homogeneous of degree one in q. Denote by $J(T, A)$ the Weierstrass integral of F over the surface represented by T [ibid., Appendix B]. Let $(\mathscr{M}, \mathscr{M}^*, f, s)$ be an unrestricted factorization of T, as

described in the review above, such that $s\mathscr{M}^*$ is a subset of X; give C, r_C, G_C, $\mathscr{K}$ the meanings assigned in that review. It is shown that $J(T, A)=\sum J(sr_C/, G_C)$, $C \in \mathscr{K}$. *P. V. Reichelderfer* (Columbus, Ohio).

Silverman, E. Morrey's representation theorem for surfaces in metric spaces. Pacific J. Math. **7** (1957), 1677–1690.

The author considers continuous mappings (surfaces) T from a disc R into a metric space E, or $T: p=p(w)$, $w=(u, v) \in R$, $p \in E$. The Lebesgue area of T had been previously defined by the author so as to be invariant both with respect to Fréchet equivalences in R and with respect to isometrisms between E and other metric spaces E'. Also, the Lebesgue area of T reduces to the usual Lebesgue area if E is Euclidean. Thus it is not restrictive to suppose that E is the space m of bounded sequences, i.e., $p=\{x^i\}$, $|x^i|<M$, with distance $\|p, q\|$ of p and $q=\{y^i\}$ given by $\sup|x^i-y^i|$. Thus $T: p=p(w)=\{x^i\}$. The following perspicuous definition is given. A mapping T is said to be quasi conformal provided (a) each component $x^i=x^i(w)$, $w \in R$, is absolutely continuous in the sense of Tonelli (ACT) in R, (b) the first partial derivatives x_u^i, x_v^i (a.e. in R) are L^2-integrable in R, and (c)

$$\sup_i [(x_u^i)^2+(x_v^i)^2]=\sup_{i,k}[x_u^i x_v^k-x_v^i x_u^k]$$

a.e. in R. By making use of a corresponding concept of Dirichlet integral (D-integral) the following main theorem is proved: Every open nondegenerate surface T of finite Lebesgue area has a quasi conformal representation T' on R (i.e., T' and T are Fréchet equivalent) for which the Lebesgue area is given by the corresponding D-integral. The proof is based, to a great extent, on the previous theory of the author [Riv. Mat. Univ. Parma **2** (1951), 47–76; MR **13**, 122], on the extension to the space m of the variational technique used by the reviewer for the proof of a Schwarz-type representation theorem [Ann. Scuola Norm. Sup. Pisa (2) **12** (1943), 61–84; MR **8**, 142] and by a reasoning of C. B. Morrey [Amer. J. Math. **55** (1933), 683–707]. *L. Cesari* (Baltimore, Md.).

Szekeres, G.; and Binet, F. E. On Borel fields over finite sets. Ann. Math. Statist. **28** (1957), 494–498.

See also: Partial Order, Lattices: Haupt and Pauc. **Functions of Real Variables:** Rajagopal.

Functions of Complex Variables

Weiner, L. M.; and Porcelli, Pasquale. A derivation of Cauchy's inequality for polynomials. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 25–27.

The principal object of this note is to derive Cauchy's inequality for the coefficients of a power series by using properties of the roots of unity. (From the authors' summary.) *A. G. Azpeitia* (Providence, R.I.).

Pontryagin, L. S. On the zeros of some transcendental functions. Amer. Math. Soc. Transl. (2) **8** (1958), 19–20.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 1279–1280 [MR **16**, 23].

Clunie, J. On a paper of Kennedy. J. London Math. Soc. **33** (1958), 118–120.

The theorem in question concerns n Jordan arcs C_k joining 0 to ∞ and nonintersecting elsewhere, dividing the plane into n disjoint regions D_k. Let u_k be subharmonic in D_k and such that $\limsup u_k(z) \leq 0$ as z approaches the boundary of D_k, while $u_k(z)>0$ somewhere in D_k. Let $\sigma_k(r)=\sup u_k(z)$ for $z \in D_k$, $|z|=r$; let $\sigma(r)=\max \sigma_k(r)$, $1 \leq k \leq n$. Kennedy stated [Proc. London Math. Soc. (3) **5** (1955), 22–47; MR **16**, 809] that $\liminf \sigma(r)r^{-1/n}<\infty$ implies $\lim(\arg z)(\log|z|)^{-1}=0$ as $z \to \infty$ on C_k, but his proof was incomplete. He later gave a proof under an additional hypothesis [ibid. **6** (1956), 518–547; MR **18**, 647]. The author shows that the theorem is correct in its original form. *R. P. Boas, Jr.* (Evanston, Ill.).

Albrecht, Rudolf. Einige Ungleichungen für in einem Parallelstreifen holomorphe Funktionen. Monatsh. Math. **62** (1958), 146–162.

The functions concerned have restricted real parts. The results are simple consequences of Pick's lemma and the elementary mappings of the unit circle onto a strip. *W. K. Hayman* (London).

★ **Springer, George. Introduction to Riemann surfaces.** Addison-Wesley Publishing Company, Inc., Reading, Mass. 1957. viii+307 pp. $9.50.

The theory of Riemann surfaces is experiencing a strong revival. An indication of this is the number of books appearing in the field. None were published during the four decades following Weyl's classic [Die Idee der Riemannschen Fläche, Teubner, Leipzig-Berlin, 1913], but the last five years have given birth to ten books largely devoted to this theory. This awakened interest has created a need for an introductory text in English. The book under consideration fills this gap.

The book is written specifically with graduate (and advanced undergraduate) students in mind. There are no prerequisites beyond standard first courses in complex variables, real variables, and algebra. What is needed of topology and Hilbert space theory is derived from the beginning. Concepts and theorems are illuminated by examples and excellent figures, proofs are clarified by heuristic remarks, and the inventiveness of even the good student is challenged by a well chosen problem collection. The style, while very readable, never becomes "insultingly simple" and even the specialist can derive pleasure from reviewing basic material in a well organized form.

Regarding closed Riemann surfaces, the title "introduction" is modest: these surfaces have received, in the largest chapter of the book, a quite comprehensive treatment. It is true that this part of the theory reached a rather final shape decades ago and that no new presentation can add much. Also, the modern doctrine of abstract Riemann surfaces could be approached without knowledge of algebraic functions, even without the foggiest idea of the Riemann surface of $\sqrt{z}$. But the pedagogical value of such "concrete" surfaces is undeniable, the beautiful classical theory being particularly suitable for developing the student's ability of geometric thinking. Furthermore, the theory on differentials on closed Riemann surfaces is a good starting point for modern literature on the space of Riemann surfaces and the Riemann-Roch theorem on complex manifolds.

As to open surfaces, the author informs us that the book "is not meant to be a survey of the current work being done in the realm of Riemann surfaces". Specifically, he restricts his attention to topological aspects and the uniformization theory, which he treats with rigor, using modern tools from combinatorial topology and orthogonal

projection. Constructive methods would lead somewhat quicker to a more general function theory and related extremal problems, but a knowledge of Hilbert space methods on Riemann surfaces is a must, and the uniformization is as good a context as any to introduce it. It also was time that someone incorporated in a text the elegant mechanism of Cartan's exterior differential calculus, which should be a tool of every graduate student.

Concerning earlier literature the author acknowledges the influence of Weyl's 1913 book and of lectures delivered by Ahlfors at Harvard University in 1948. The former perhaps contributes primarily to the general set-up of the book, the latter to chapters dealing with topology and exterior differential calculus.

The book is, roughly speaking, divided into two equal parts, the first (Ch. 1–5) devoted to Riemann surfaces, the second (Ch. 6–10) to functions and differentials. The introductory Ch. 1 is largely heuristic and presents main goals, methods, and concepts of the book. Ch. 2, General Topology, is a brief introduction to Hausdorff spaces, their subsets and mappings, and to the concepts of manifold and abstract Riemann surface. In Ch. 3, Riemann Surface of an Analytic Function, the author gives the definitions of complete analytic function, the corresponding analytic manifold, the analytic configuration, and its geometric counterpart, the Riemann surface of an analytic function. The last is shown to be a special case of an abstract Riemann surface. Ch. 4, Covering Manifolds, is a discussion of smooth and unlimited covering manifolds, the monodromy theorem, the fundamental group, and covering transformations.

Ch. 5, Combinatorial Topology, is given over to global properties of triangulable manifolds. Barycentric coordinates and subdivision are introduced, and the orientability of manifolds is considered. It is shown that every Riemann surface is orientable and that every orientable triangulable manifold can be made into a Riemann surface by defining an analytic structure on it. Differentiable and analytic curves are introduced and their properties briefly discussed. Every compact orientable surface is shown to be homeomorphic to a sphere with g handles. The genus g is then characterized by homology groups and Betti numbers, and relations are given between the fundamental group and the first homology group. The chapter ends with an explicit computation of the homology groups on compact surfaces.

Ch. 6, Differentials and Integrals, is an exposition of the usual material centering around Stokes' theorem. An introduction to exterior differential calculus is given and harmonic and analytic differentials briefly dealt with. Ch. 7, Hilbert Space on Differentials, introduces smoothing operators, Weyl's lemma, and orthogonal projections, to be used in the existence proofs of Ch. 8. In the latter chapter, the existence of exterior harmonic differentials is established, with given singularities and a finite norm over a boundary neighborhood. It is also shown that every Riemann surface has a countable basis. The existence theorems of Ch. 8 are then used in Ch. 9 to establish the parallel slit mapping and the general uniformization theorems. Automorphic functions and non-Euclidian geometry on the uniformizing disk are considered, and every Riemann surface is shown to be triangulable.

Ch. 10, Compact Riemann Surfaces, is an account of bilinear relations, the Riemann-Roch theorem, Weierstrass points, Abel's theorem, the Jacobi inversion problem, and the field of algebraic functions. For illustration, the hyperelliptic case is treated in detail.

L. Sario (Los Angeles, Calif.).

Boas, R. P., Jr.; and Schaeffer, A. C. New inequalities for entire functions. J. Math. Mech. 7 (1958), 191–205.

Let $f(z)$ be regular for $x\geqq 0$ ($x=\mathrm{Re}\, z$) and of exponential type $\tau(<\pi)$. If $\lim_{n\to\infty} f(n)$ exists, it was shown by Plancherel and Pólya that $\lim_{x\to+\infty} f'(x)=0$. The converse of this statement is false. One of the objects of the authors is to obtain generalizations of the result of Plancherel and Pólya which have valid converses. In this direction the authors prove: let $f(z)$ be regular and of exponential type $\tau(<\pi)$ in the half-plane $x>0$; then $f(x)$ is bounded for real $x(>1)$ (or approaches a limit as $x\to+\infty$) if and only if $\sum_1^\infty f'(k)$ has bounded partial sums (or converges).

In the case of entire functions of exponential type π, it is possible to assert more. The authors also investigate the possibility of replacing convergence by summability and obtain a result involving $(C, 1)$-summability.

The authors prove other results along the same general lines; one of the simplest will be stated explicitly: if $f(z)$ is regular and of exponential type $\tau(<\pi)$ in the half-plane $x>0$, then

$$\sum_{n=1}^{\infty} |f(n+1)-f(n)|^p$$

converges ($p>0$) if and only if $\sum |f'(n)|^p$ converges.

A. Edrei (Syracuse, N.Y.).

Gaier, Dieter. Über ganze Funktionen vom Exponentialtyp mit Lückenreihen. Math. Z. 68 (1958), 488–497.

For functions $f(z)$ which are regular and of exponential type in an angle $|\arg z|\leqq\alpha\leqq\frac{1}{2}\pi$, Carlson, Pólya, Cartwright and others have obtained relations between the behavior of f on the positive integers and its behavior on the positive real axis. The author extends such results to the case where the usual type restrictions are violated. Suppose that $\limsup (1/n)\log|f(n)|\leqq 0$. Denote $\limsup (1/r)\log|f(re^{i\theta})|$ by $h(\theta)$. The following are samples of the results proved. (i) If $\alpha<\frac{1}{2}\pi$, $h(\pm\alpha)\leqq a\sin\alpha$, then $h(0)\leqq \max\{0, (a-\pi)\tan\alpha\}$. (ii) If $\alpha=\frac{1}{2}\pi$ and $f(z)$ is of exponential type $k\geqq\pi$, then $h(0)\leqq(k^2-\pi^2)^{\frac{1}{2}}$. In the following, let $f(z)=\sum a_k z^{n_k}$ be entire, and suppose that the sequence $\{n_k\}$ has maximum density $\leqq\Delta<\frac{1}{2}$. (iii) If $f(z)$ is of exponential type $<\pi/\sin\pi\Delta$ in $\mathrm{Re}\, z\geqq 0$, then $f(z)$ is of exponential type $\leqq 0$ there. (iv) If $\Delta=0$ and $f(z)$ is of finite type in some angle $|\arg z|\leqq\alpha$, then $f(n)\sim e^{cn}$ ($c\geqq 0$) implies $f(x)\sim e^{cx}$. The author applies (iv) to Borel summability of gap series.

J. Korevaar (Madison, Wis.).

Hiong, King-Lai. Sur la limitation d'une fonction holomorphe sans zéro et admettant une valeur exceptionnelle B. Bull. Sci. Math. (2) 81 (1957), 71–77.

It is well known that a simple application of the methods of R. Nevanlinna leads to a proof of Schottky's theorem giving, in terms of $|z|$ and $|f(0)|$, an upper bound of the maximum modulus of a function $f(z)$ which is regular, does not vanish and does not take the value 1 in the disk $|z|<1$. The author shows that the latter condition may be replaced by the following one: the value 1 is exceptional in the sense of Borel, that is

$$\limsup_{r\to 1}\frac{N(r, 1)}{\log(1/(1-r))}<+\infty.$$

($N(r, 1)$ is one of the usual symbols of Nevanlinna's theory.)

A. Edrei (Syracuse, N.Y.).

Noshiro, K. On the theory of cluster sets of analytic functions. Amer. Math. Soc. Transl. (2) **8** (1958), 1–12. Translated from Sûgaku **5** (1953), 65–72 [MR **15**, 614].

Tanaka, Chuji. An extension of Kintchine-Ostrowski's theorem and its applications. Kôdai Math. Sem. Rep. **9** (1957), 97–104.

The author proves three theorems, of which the two simpler ones are as follows. Th. 1: Let $\{f_n\}$ be a sequence of analytic functions in $|z|<1$, such that

$$(*) \quad \int_0^{2\pi} \log^+ |f_n(re^{i\theta})|d\theta < M \quad (0\leq r<1, n=1, 2, 3, \cdots)$$

for some finite constant M, and suppose that the sequence $\{f_n(e^{i\theta})\}$ converges on a set of positive measure. Then $\{f_n(z)\}$ converges uniformly on every compact subset of $|z|<1$. (The author's contribution is to weaken the hypothesis of uniform boundedness to (*).) Th. 3: Suppose $f(s)$ is analytic in the strip S: $\alpha<\sigma<\beta$ $(s=\sigma+it)$, $p>0$, h is a harmonic function in S, and $|f(s)|^p\leq h(s)$ in S. Suppose further that $\{h(\sigma_0+nit_0)\}$ is bounded for some σ_0+it_0 in S and for $n=0, 1, 2, \cdots$, that f is continuous on $\sigma=\alpha$ (except at ∞), and that $\lim_{t\to+\infty} f(\alpha+it)=a$ $(\neq\infty)$. Under these conditions, $f(s)$ tends uniformly to a as $t\to+\infty$, in the strip $\alpha\leq\sigma\leq\beta-\varepsilon$, for any $\varepsilon>0$. (In the classical analogue of this (Montel), f is assumed to be bounded.) *W. Rudin* (New Haven, Conn.).

Pokalo, A. K. Summation of $B^{(r)}$ functions. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 750–753. (Russian)

The class $B^{(r)}$ consists of all functions analytic in the open unit disk and satisfying $|f^{(r)}(z)|\leq 1$ there. The author's theorem gives a complicated formula for certain weighted partial sums $\sum_0^n \mu_{nk}c_kz^k$ of the Taylor series $\sum c_kz^k$ for f. Stečkin proved the $(C, 1)$ case of this formula in Izv. Akad. Nauk SSSR. Ser. Mat. **17** (1953), 461–472 [MR **15**, 206]. *H. Mirkil* (Hanover, N.H.).

Krasnoščekova, T. I. Certain properties of series in Faber polynomials. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 38–41. (Russian)

The author transfers two propositions from the theory of Taylor series to the theory of Faber series. One of them is Pólya's theorem on overconvergence (in the domain of existence of the corresponding function) of a Taylor series $\sum a_nz^{k_n}$ with $\limsup k_{n+1}/k_n=\infty$. Since the paper was submitted, a more general result by Tietz has been published [Michigan Math. J. **4** (1957), 175–179, Th. 3; MR **19**, 1045]. *G. Piranian* (Ann Arbor, Mich.).

Eweida, M. T. On the three-point expansions. Bull. Coll. Arts Sci. Baghdad **2** (1957), 92–95.

The author considers the following problem: Given three increasing sequences of integers $p_1, p_2, \cdots$; $q_1, q_2, \cdots$; $r_1, r_2, \cdots$; what conditions must they satisfy in order that a knowledge of the numbers $f^{(p_n)}(a_1)$, $f^{(q_n)}(a_2)$, $f^{(r_n)}(a_3)$, $n=1, 2, 3, \cdots$, and a_1, a_2, a_3 all different, may determine $f(z)$? Following J. M. Whittaker [Proc. London Math. Soc. (2) **36** (1933), 451–469], he defines the notion of a standard set of polynomials in relation to the sequences (p, q, r), and also the idea of completeness of (p, q, r). He then proves necessary and sufficient conditions that (p, q, r) be complete. *J. G. Herriot* (Stanford, Calif.).

Thimm, Walter. Die Struktur der Menge der singulären Bildpunkte einer meromorphen Abbildung. Math. Ann. **134** (1957), 143–153.

L'objet du mémoire est l'étude de l'image $M(x)$ donnée par une transformation méromorphe en un point d'indétermination x, et de la dépendance de $M(x)$ par rapport à x. L'application méromorphe $z=F(x)$ est supposée définie par $z_j=F_j(x_1, \cdots, x_n)$ $(1\leq j\leq k)$, où les F_j sont des fonctions méromorphes dans un voisinage de l'origine $x=0$; z est considéré dans l'espace $\bar{C}_z^k$, produit des k plans complexes fermés. On savait déjà par les travaux de l'auteur [Math. Z. **57** (1953), 456–480; MR **14**, 971] que l'ensemble $M(x)$ des valeurs limites de $F(x')$ quand $x'\to x$, x étant un point d'indétermination, est un ensemble algébrique. Le résultat essentiel obtenu à présent s'énonce: si F_Δ est la restriction de F à un germe Δ d'ensemble analytique de dimension complexe m, irréductible à l'origine $(x=0)$, si A est l'ensemble des points d'indétermination de F_Δ sur Δ, il existe dans un voisinage U_x de l'origine une décomposition finie, $A=\sum_1^l \beta_p$, avec les propriétés suivantes: les ensembles β_p sont obtenus à partir d'une suite finie α_p d'ensembles analytiques dans U_x, irréductibles à l'origine, en opérant par différence

$$\beta_p=\alpha_p-\gamma_p,$$

γ_p étant la réunion de certains α_q de dimension inférieure, contenus dans α_p. Il existe d'autre part des polynomes en nombre fini $P_\tau(z_1, \cdots, z_k, x_1, \cdots, x_n)$, des z_j, à coefficients holomorphes des x_i dans U_x, de manière que, quand x varie sur β_p, $M(x)$ soit exactement l'ensemble algébrique obtenu en annulant les P_τ et, de plus, conserve une dimension δ_p constante $(\sigma_p+\delta_p\leq m-1, \sigma_p$ étant la dimension de $\alpha_p)$. Le graphe de F dans $U_x\times\bar{C}_z^k$ est donc formé des zéros d'un idéal dans l'anneau des polynomes des z_j à coefficients holomorphes de x à l'origine; ce résultat est relié aux profonds travaux de l'auteur sur les idéaux du même type [0-ensembles, cf.: J. Reine Angew. Math. **193** (1954), 183–208; MR **17**, 409]; un paragraphe du présent travail donne une présentation plus simple de certains énoncés antérieurs. *P. Lelong* (Paris).

See also: Linear Algebra: Parodi. **Special Functions:** Rankin.

Harmonic Functions, Convex Functions

See: Functions of Complex Variables: Clunie. **Approximations, Orthogonal Functions:** Vzorova. **Topological Vector Spaces:** Fan. **Complex Manifolds:** Lelong. **Fluid Mechanics, Acoustics:** Bloomer.

Special Functions

Al-Salam, W. A.; and Carlitz, L. Some finite summation formulas for the classical orthogonal polynomials. Rend. Mat. e Appl. (5) **16** (1957), 74–95.

A host of finite summation analogs for integral formulas involving Legendre, Jacobi, Hermite, and Laguerre polynomials and Bessel functions are obtained. Most of these are completely new, but some are generalizations of known results. At the basis of the systematic method used is the evaluation of certain finite trigonometric sums; for example,

$$\frac{1}{t}\sum_{k=0}^{t-1} \cos^p\left(\zeta+\frac{k\pi}{t}\right)\cos q\left(\zeta+\frac{k\pi}{t}\right)=\frac{1}{\pi}\int_0^\pi \cos^p\theta\cos q\theta d\theta$$

provided $t>p+q$ and $p+q$ is even. If $f(z)$ is a polynomial of degree less than or equal to n, then $f[\cos^2(\zeta+k\pi/t)]$ and $f(\cos^2\theta)$ are put in place of $\cos^p(\zeta+k\pi/t)$ and $\cos^p\theta$ above,

and the resulting integral formula is interpreted to yield conclusions in terms of special functions.

N. D. Kazarinoff (Ann Arbor, Mich.).

Al-Salam, W. A.; and Carlitz, L. Congruence properties of the classical orthogonal polynomials. Duke Math. J. **25** (1957), 1–9.

Congruence relations are obtained which are analogs of the authors' finite summation analogs of certain integral formulas for Hermite, Jacobi, Legendre, and Laguerre polynomials [see the above review]. *N. D. Kazarinoff.*

Al-Salam, W. A.; and Carlitz, L. Finite summation formulas and congruences for Legendre and Jacobi polynomials. Monatsh. Math. **62** (1958), 108–118.

The authors first establish the formula

$$\{x(1+\cos\Phi)+i\sin\Phi\}^n=$$

$$P_n(x)+\sum_{k=1}^{n}\frac{n!}{(n+k)!}\,(X^{ik}e^{ik\Phi}+X^{-ik}e^{-ik\Phi})P_n^{k}(x),$$

where $X=(x+1)/(x-1)$, and derive an expression for

$$t^{-1}\sum_{s=0}^{t-1}\{x(1+\cos\alpha_s)+i\sin\alpha_s\}^n\exp(ir\alpha_s),$$

where $\alpha_s=\Phi+2s\pi/t$, as a finite sum of functions of the form $X^{\beta}P_n^{\mu}(x)e^{ik\Phi}$, with appropriate β, μ. Results are also given relating to congruences of such expressions modulo an arbitrary prime.

Analogous formulae are found for the Jacobi polynomials by expanding $(1+xe^{i\phi})^{\mu}(1+xe^{-i\phi})^{\nu}$, then for certain polynomials in several variables by expanding $(x\cos\alpha+y\cos\beta+z\cos\gamma)^n$. Many special cases and subsidiary results are given. *F. M. Arscott.*

Al-Salam, Waleed A. Some remarks on the Turan expression. Bull. Coll. Arts Sci. Baghdad **2** (1957), 104–111.

Given a sequence of functions $\{f_n\}$, its corresponding nth Turan expression $T_n(f)$ is $f_n^2-f_{n+1}f_{n-1}$. Various lower bounds are known for $T_n(f)$ when f_n is a cylinder function or one of the classical polynomials. Conditions are given under which a given form of $T_n(f)$ characterizes the sequence $\{f_n\}$. The conditions the functions h_n, p_n, and q_n must satisfy in order that a solution of the difference equation $h_nu_{n+1}+p_nu_n+q_nu_{n-1}=0$ should satisfy the Turan inequality $T_n(u)\geqq 0$ are found. *N. D. Kazarinoff.*

Rankin, R. A. The differential equations associated with the uniformization of certain algebraic curves. Proc. Roy. Soc. Edinburgh. Sect. A. **65** (1958), 35–62.

If $G(z, u)$ is a polynomial in z and u and $G(z, u)=0$ is of genus p, then the problem of uniformization is solved for $p=0$, 1, but open for $p\geqq 2$, even in the general hyperelliptic case $u^2=\prod_{r=1}^{2p+2}(z-e_r)=g(z)$. It is known [Ford, Automorphic functions, McGraw-Hill, N.Y., 1929, p. 240] that the uniformization may be effected by functions $z=z(t)$, $u=u(t)$, where $z(t)$ and $u(t)$ are both automorphic, belonging to the same Fuchsian group of the first kind Γ. If Γ is given, then one knows how to construct the functions automorphic under Γ. In the hyperelliptic case Γ is a subgroup of finite index of the monodromy group Γ_0 of the differential equation $d^2\eta/dz^2+R(z)\eta=0$, where the rational function $2R(z)=\{t, z\}=(t'''/t')-\frac{3}{2}(t''/t')^2$, the Schwarzian derivative. The author generalizes this result to the case of equations $u^m=\prod_{r=1}^{n}(z-e_r)$. As t is an unknown function, $R(z)$ has to be determined by considering the behaviour of $\{t, z\}$ at the branch points and at infinity. In the hyperelliptic case, $R(z)=(3/16)\{\sum_{r=1}^{2p+2}(z-e_r)^{-2}-h(z)/g(z)\}$, where $h(z)$ is a polynomial of degree $2p$, containing $2p-1$, a priori unknown, accessory parameters. They are uniquely determined by the fact that Γ_0 is Fuchsian, but no general method for their computation seems known. E. T. Whittaker's conjecture [J. London Math. Soc. **4** (1929), 274–278] that

$$R(z)=W(z)\equiv\frac{3}{16}\left\{\left(\frac{g'}{g}\right)^2-\frac{2p+2}{2p+1}\frac{g''}{g}\right\}$$

had been proven in the cases $u^2=z^{2n+1}+1$ and $u^2=z^{2n}-1$. The author proves Whittaker's conjecture for equations of the form $u^2=g(z)$, where $g(z)$ is either $z^{p_1}(z^N-1)$ ($p_1=0$ or 1, $N\geqq 2$), or any one of 18 functions, explicitly indicated; these are simple combinations of the ground forms for the polyhedral groups, but too complicated to be quoted here. In case $g(z)=z^{p_1}\prod_{i=}^{r}(z^N-\lambda_i)$ ($p_1=0$, 1, $r\geqq 2$, $N\geqq 2$, $\lambda_i\neq\lambda_j$ for $i\neq j$ $\lambda_i\neq 0$), and $=g(z)$,

$$R(z)=W(z)+z^{N-2}\prod_{i=0}^{r}(z^N-\lambda_i)^{-1}\sum_{m=0}^{r-2}a_mz^{mN},$$

where the a_m ($m=0, 1, \cdots, r-2$) are $r-1$ still undetermined constants. For more general forms of $g(z)$, it is shown that $R(z)=W(z)+q(z)/g(z)$, where $q(z)$ is a polynomial in the mentioned ground forms and contains r unknown constants. The form of $R(z)=F(z)$ is determined also for $u^m=z^p(z^N-1)^r$ ($0\leqq p<m$, $m\geqq 2$, $r\geqq 1$, $N\geqq 2$, $(m, p, r)=1$), as well as for $u^m=g(z)$, provided that $g(z)$ is one of a large class of explicitly indicated functions. In the hyperelliptic case ($m=2$), $F(z)$ reduces to Whittaker's function $W(z)$. The author then proceeds to solve the differential equations $R(z)=\frac{1}{2}\{t, z\}=F(z)$ in many of the cases for which he has established it. As a consequence, he succeeds in obtaining the uniformizing functions for the algebraic equations $u^m=z^p(z^N-1)^r$ and many others. In all cases both u and z are simple algebraic functions of the Riemann-Schwarz triangle function $S(\nu_1, \nu_2, \nu_3; t)$ that maps a curvilinear triangle of angles $(\pi/\nu_1, \pi/\nu_2, \pi/\nu_3)$ onto the upper half-plane.

E. Grosswald (Philadelphia, Pa.).

Rankin, R. A. The construction of automorphic forms from the derivatives of given forms. Michigan Math. J. **4** (1957), 181–186.

If $f_i(z)$ ($i=0, 1, \cdots, m$) are $m+1$ given, meromorphic automorphic forms of real or complex dimensions $-k_i$ with multipliers v_i and belonging to some horocyclic group Γ, one writes, as usual, $f_i\in\{\Gamma, k_i, v_i\}$ ($i=0, 1, \cdots, m$). The author determines those polynomials P in the f_i's and their derivatives which are automorphic forms of z with respect to Γ. This is a generalization of the author's previous results [J. Indian Math. Soc. (N.S.) **20** (1956), 103–116; MR **18**, 571], where the case $m=0$ is settled. In order to be automorphic, P has to satisfy a certain elementary necessary condition which is too complicated to be quoted; the polynomials that satisfy it are called admissible. Next, set $h_{i\nu}=f_i^{(\nu)}/\Gamma(k_i+\nu)\nu!$, $c_{i\nu}=\psi_{i\nu}/h_{i0}^{\nu}$, with $\psi_{i\nu}=(-1)^{\nu-1}\|a_{pj}\|$; where the determinant $\|a_{pj}\|$ has the elements $a_{1j}=jh_{ij}$, and, for $p\geqq 2$, $a_{pj}=h_{i,p-j+1}$ if $p-j+1\geqq 0$, $a_{pj}=0$ otherwise. For $i\neq 0$ set also $w_i=k_0f_0f_i'-k_if_0'f_i$, $t_i=w_i/(f_0f_i)$. Then, if none of the k_i's is a non-positive integer, the main result of the paper may be formulated as follows: If P is admissible in the f_i's and their derivatives, then $P(z)\in\{\Gamma, k', v'\}$ if and only if P is a polynomial in the f_i, $c_{i\mu}$ ($\mu\geqq 2$) and t_i. This polynomial can be written as a polynomial in f_i, $1/f_i$, $\psi_{i\mu}$ ($\mu\geqq 2$) and w_i. The values of k' and v' are indicated and the case when some, or all, of the k_i's are-non-positive

integers is also discussed. Coming back to the case of a single integral automorphic form f, let $f \in \{\Gamma, k, v\}$, where Γ is a zonal horocyclic group, and let f have a zero of order $m(>0)$ at infinity. The author shows that f satisfies a homogeneous differential equation of order n_k. Here n_k is the least value of ν for which $\hat{p}(\nu) > N_\nu$, where $\hat{p}(\nu)$ stands for the number of partitions of ν into parts not less than 2 and $N_\nu = N_\nu(\Gamma, k, v, m)$ is the number of linearly independent integral forms of $\{\Gamma, \nu(k+2), v^\nu\}$, having zeros of order at least νm at infinity. The functions $f(z) = \Delta(z)G_{k-12}(z)$ ($k=12, 16, 18, \cdots, 38$) are considered as illustrations. *E. Grosswald* (Philadelphia, Pa.).

Oberhettinger, F. On the derivative of Bessel functions with respect to the order. J. Math. Phys. **37** (1958), 75–78.

Closed expressions in terms of the exponential integral and sine and cosine integrals are established for the first ν-derivatives of the functions $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ when $\nu = \pm\frac{1}{2}$. *F. W. J. Olver* (Washington, D.C.).

Palm, Enok. On the zeros of Bessel functions of pure imaginary order. Quart. J. Mech. Appl. Math. **10** (1957), 500–503.

It is proved that no non-trivial solution of the differential equation $z^2u'' + zu' + (\nu^2 - z^2)u = 0$ can have more than one zero with prescribed imaginary part. This result has an application in the theory of stability of Couette flow. *F. W. J. Olver* (Washington, D.C.).

Orts, J. M.[a]. Analytic representation of hypergeometric functions by series of polynomials. Collect. Math. **9** (1957), 145–151. (Spanish)

In Euler's integral

$$F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-tx)^{-a} t^{b-1}(1-t)^{c-b-1}\,dt,$$

the author substitutes the known expansion of $(1-tx)^{-a}$ in a series of Legendre polynomials of variable tx and thus obtains an expansion of $F(a, b; c; x)$ in a series of certain hypergeometric polynomials. *A. Erdélyi.*

Agarwal, Km. Nirmala. Extensions of certain results on partial sums of hypergeometric type. Ganita **7** (1956), 139–143.

R. P. Agarwal has derived transformations of truncated basic hypergeometric series into constant multiples of other such series [Proc. Cambridge Philos. Soc. **49** (1953), 441–445; MR **15**, 122]. The author's results are of the same type but concern double series similar to those defined by Chaundy. *N. D. Kazarinoff.*

Ragab, F. M. On the product of two confluent hypergeometric functions. Monatsh. Math. **61** (1957), 312–317.

The integral $\int_0^\infty x^{-1}\,{}_1F_1(a; c; -x)\,{}_1F_1(a; c; -b/x)\,dx$ is evaluated by specializing a known result involving E functions. A generalization of this integral to a product of multiple integrals is also evaluated.

N. D. Kazarinoff (Ann Arbor, Mich.).

Hochstadt, Harry. Addition theorems for solutions of the wave equation in parabolic coordinates. Pacific J. Math. **7** (1957), 1365–1380.

These are the results announced and reviewed earlier [Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-18** (1956); MR **17**, 1084].

B. S. Popov (Skopje).

Ferrer Figueras, Lorenzo. On the partial differential equation $\sum_{\omega=0}^{s}\sum_{r=0}^{\infty} A_{\omega r} z^r \partial^{\omega+r}u/\partial x^{\omega-r}\partial z^r = 0$ corresponding to the generating functions $u_{q,k} = [1 - x^q + (x-z)^q]^{-1/k}$. Collect. Math. **9** (1957), 77–85. (Spanish)

In an earlier paper [Collect. Math. **6** (1953), 221–291, in particular p. 267; MR **16**, 695] the author remarked that as a consequence of the differentiation formulas satisfied by the polynomials generated by the expansion of $u_{q,k}$ in powers of z, the latter function satisfies some differential equation of the form given in the title. He now obtains the relations from which the coefficients $A_{\omega,r}$ can be computed and proves that the order s of this differential equation is the least positive integer satisfying

$$(s-1)(s-q) - \binom{s-2}{2} + 3 - q > 0.$$

A. Erdélyi (Pasadena, Calif.).

See also: Numerical Methods: Belousov. **Optics, Electromagnetic Theory, Circuits:** Körper. **Quantum Mechanics:** Shepanski and Butler.

Sequences, Series, Summability

★ **Маркушевич, А. И. [Markuševič, A. I.] Ряды. Элементарный очерк. [Series. Elementary outline.] 3rd ed., revised and augmented.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 187 pp. 3 rubles.

A semi-popular account, accessible to readers who have not yet studied calculus. Many interesting details are presented from the history of mathematics.

Davis, Chandler. A device for studying Hausdorff moments. Trans. Amer. Math. Soc. **87** (1958), 144–158.

Let I be a subset of the non-negative integers and $\{\mu_n; n \in I\}$ a set of real numbers. It is well known that a necessary and sufficient condition for

$$\mu_n = \int_0^1 t^n\,d\Phi(t), \quad d\Phi \geqq 0,$$

is that there exists a matrix A such that $0 \leqq A \leqq 1$ and $\mu_n = (A^n)_\infty$ ($n \in I$). The matrix A may be chosen to be a Jacobi matrix. The object of this paper is to give a convenient canonical form for the Jacobi matrix A.

The author proves his main theorem on the canonical form for the case where each μ_n is a bounded self-adjoint operator on a Hilbert space and the entries of A are bounded operators on this space. The actual statement of the theorem is too lengthy to be included here.

The last part of the paper examines the classical (numerical) Hausdorff moment problem in the light of the canonical form obtained for A. *A. Devinatz.*

Andersen, A. F. On the extensions within the theory of Cesàro summability of a classical convergence theorem of Dedekind. Proc. London Math. Soc. (3) **8** (1958), 1–52.

Die Arbeit befasst sich mit dem hinreichenden Teil des bekannten Satzes von I. Schur über Summierbarkeitsfaktoren und mit Verallgemeinerungen dieses Satzes. In dieser Richtung bewies Bosanquet (im wesentlichen) folgenden Satz [dieselben Proc. (2) **50** (1949), 482–496; MR **10**, 368]: Ist (1) $S_\nu^r(u_\nu) = \sum_{\mu \leq \nu} A_{\nu,-\mu}{}^{r-1} u_\mu = o(\nu^{r+\delta})$, (2) $\varepsilon_\nu = O(\nu^{-\rho-\delta})$, (3) $\sum \nu^{r+\delta}|\Delta^{r+1}\varepsilon_\nu| < \infty$ ($0 \leqq \rho \leqq r$, $\delta \geqq 0$), so ist $\sum_{\nu=0}^{\infty} u_\nu\varepsilon_\nu$ $(C, r-\rho) = \sum_{\nu=0}^{\infty} S_\nu^r(u_\nu)\Delta^{r+1}\varepsilon_\nu$. Für diesen Satz wird zunächst ein neuer Beweis gegeben. Dabei werden

Spezialfälle dieses Satzes (r ganz; $r=\rho$, d.h. der Bohr-Hardysche Satz) im Hinblick auf möglichst einfache Beweisanordnung eingehend diskutiert. Der Verf. zeigt dann, dass aus den obigen Voraussetzungen über u_ν und ε_ν für $0\leqq\sigma\leqq r+1$ die Beziehung $\sum_{\nu=0}^{\infty} S_\nu^{\sigma-1}(u_\nu)\Delta^\sigma\varepsilon_\nu\,(C,\alpha)=\sum_{\nu=0}^{\infty} S_\nu^{r}(u_\nu)\Delta^{r+1}\varepsilon_\nu$ mit $\alpha=r-\rho$ $(0\leqq\sigma\leqq\rho)$, $\alpha=r-\sigma$ $(\rho<\sigma\leqq r)$, $\alpha=0$ $(r<\sigma\leqq r+1)$ abgeleitet werden kann. Man kann hier noch einen Schritt weitergehen und die Summierbarkeit von $\sum s_\nu^{\sigma-1}(u_\nu)\Delta^\sigma\varepsilon_\nu$ untersuchen. Im Rahmen eines Referates ist es nicht möglich, auf die vielen weiteren Ergebnisse einzugehen. Es handelt sich hauptsächlich um die Frage nach der Zulässigkeit negativer Werte von t in den Voraussetzungen, ferner um Ergebnisse bei Abänderung der Symbole O und o in den Voraussetzungen (1) und (2); wird in beiden Fällen rechts O eingesetzt, so ergibt sich die $(C,r-\rho)$-Beschränktheit der Reihe $\sum u_\nu\varepsilon_\nu$ und die $(C,r-\rho+\varepsilon)$-Summierbarkeit für $\rho>0$ und jedes $\varepsilon>0$. *A. Peyerimhoff.*

Petersen, G. M. Sequences of iterations. Math. Z. **68** (1957), 151–152.

The notion of norm introduced by Brudno [Mat. Sb. N.S. **16(58)** (1945), 191–247; MR **7**, 12] leads to many questions regarding matrix summability methods. In this note, a contribution is made toward answering one such question. It is shown that if $\{A_k\}$ is a sequence of regular matrices, whose norms form a convergent infinite product, then there is a matrix stronger (for bounded sequences) than all the iterations $A_nA_{n-1}\cdots A_1$, $n=1, 2, \cdots$. *C. Goffman* (Lafayette, Ind.).

Agnew, Ralph Palmer. The Lototsky method for evaluation of series. Michigan Math. J. **4** (1957), 105–128.

Von Lototsky [Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 61–91; MR **17**, 1074] wurde das durch die Dreiecksmatrix $a_{n\nu}=(n!)^{-1}p_{n\nu}$ $(n, \nu=1, 2, \cdots)$, $x(x+1)\cdots(x+n-1)=\sum p_{n\nu}x^\nu$, erklärte und reguläre Limitierungsverfahren eingeführt. Die Arbeit erschien in einer schwer zugänglichen Zeitschrift, der Verf. stellt daher die Lototsky'schen Ergebnisse erneut dar, zum Teil in allgemeinerer Form. Er zeigt zunächst, dass das Verfahren die geometrische Reihe (genau wie das Borel Verfahren) für $\Re z<1$ summiert und alle Euler-Knoppschen Verfahren enthält. Die durch das Verfahren limitierten Folgen genügen der Abschätzung $s_n=O(n!/(\log 2)^n)$. Ferner untersucht er die Reihen-Reihen Form des Verfahrens, seine Beziehung zum Borel Verfahren Indexverschiebung und die Wirksamkeit bei einigen speziellen Reihen. {Der Ref. stellt fest, dass Verfahren der hier behandelten Art in folgender Weise erzeugt werden können: Ist $q_n(x)=\sum_{0\leqq\nu\leqq n} q_{n\nu}x^\nu$, $q_{n\nu}\geqq 0$, $Q_n=q_n(1)\neq 0$, so ist das Verfahren $b_{n\nu}=q_{n\nu}/Q_n$ genau dann regulär, wenn $q_n(x)=o(Q_n)$ für alle $|x|<1$ gilt. Die geometrische Reihe wird summiert für alle $\Re x<1$, falls $q_n(x)=o(Q_n)$ ist für alle $\Re x<1$. Ist schliesslich noch $q_n^{(\mu)}(x)\geqq 0$ für jedes $x<0$, $\mu\geqq 0$, $n\geqq n_0(x)$, so ist das Verfahren stärker als alle Euler-Knoppschen Verfahren. Allen diesen Bedingungen genügt etwa $q_n(x)=(n^{2/3}+x)^n$.} *A. Peyerimhoff* (Giessen).

Guha, U. C. (γ, k)-summability of series. Pacific J. Math. **7** (1957), 1593–1602.

The series $\sum_0^\infty a_n$ is said to be summable (γ, k) if (i) $\sum a_n\gamma_k(nt)$ converges for $0<t<A$ and (ii) $\lim_{t\to 0}\sum a_n\gamma_k(nt)$ is finite, where $\gamma_k(x)=k/\delta^k_0\,(1-t)^{k-1}\cos xt\,dt$. The following are established. If $\sum a_n$ is summable (γ, k) where k is zero or a positive integer, then it is summable Cesàro $(C, k+\delta)$, $\delta>0$. If $\sum a_n$ is summable (γ, k), $[k]\geqq 4$, then it is summable $(C, k+\delta)$, $\delta>0$. *D. Moskovitz.*

Rajagopal, C. T. Simplified proofs of "Some Tauberian theorems" of Jakimovski: Addendum and corrigendum. Pacific J. Math. **7** (1957), 1727.

It is reported that the principal result in the papers of Jakimovski and of Rajagopal [Pacific J. Math. **7** (1957), 943–954, 955–960; MR **19**, 544] is a special case of a result of Hausdorff [Math. Z. **31** (1930), 186–196]. The author also corrects an error in his paper. *J. Korevaar.*

See also: Analytic Theory of Numbers: Richert. **Functions of Real Variables:** Rajagopal. **Functions of Complex Variables:** Gaier; Pokalo. **Trigonometric Series and Integrals:** Chen.

Approximations, Orthogonal Functions

Tchakaloff, L. Formules générales de quadrature mécanique du type de Gauss. Colloq. Math. **5** (1957), 69–73.

The author considers the following mechanical quadrature formula

$$(1)\qquad \int_a^b f(x)dx=\sum_{k=1}^{m}\sum_{\lambda=0}^{r_k} A_{k\lambda}f^{(\lambda)}(a_k)\quad(-\infty<a<b<\infty),$$

with $m>0$ and λ, $r_k\geqq 0$ designating integers, and (a_k) distinct real numbers. The problems posed are: (i) given $m>0$, $r_k\geqq 0$, $k=1, \cdots, m$, determine the largest integer N for which m abscissas a_k and M coefficients $A_{k\lambda}$ $(M=\sum_{k=1}^m (r_k+1))$ can be chosen independently of f so that (1) will be exact for all polynomials of degree $<N$; (ii) determine a_k and $A_{k\lambda}$.

It turns out that $N=\sum_{k=1}^m (2[\frac{1}{2}r_k]+2)$. In solving (ii) it is shown that $a_1, a_2, \cdots, a_m$ exist inside the interval (a, b) and are the zeros of the polynomial $P(x)=\prod_{k=1}^m (x-a_k)^{r'_k+1}$; $r_k'=2[\frac{1}{2}r_k]$, which is orthogonal over the interval (a, b) to all polynomials of degree $<m$. Also, if r_k is odd, $A_{kr_k}=0$ and, as the author has previously shown [C. R. Acad. Bulgare Sci. Math. Nat. **1** (1948), no. 2–3, 9–12; MR **10**, 743], all other $A_{k\lambda}$ may be obtained from the partial fraction decomposition of

$$R(x)=P(x)^{-1}\int_a^b \frac{P(x)-P(t)}{x-t}\,dt.$$

In particular, $R(x)=\sum_{k=1}^m \sum_{\lambda=0}^{r_k} A_{k\lambda}\cdot\lambda!(x-a_k)^{-\lambda-1}$. The interested reader should also see the paper by T. Popoviciu [Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 29–57; MR **19**, 64]. *I. Polonsky.*

Meder, J. On the estimation of Cesàro means of orthonormal series. Ann. Polon. Math. **4** (1958), 183–200.

Es sei $\{\varphi_n\}$ ein beliebiges Orthonormalsystem in $[0, 1]$. Der Verf. untersucht die Grössenordnung der Cesàromittel der Orthogonalreihe $f(x)\sim\sum a_n\varphi_n(x)$, $\sum a_n^2<\infty$ und zeig dass in $[0, 1]$ fast überall $\sigma_n^{(r)}(x)=(A_n^r)^{-1}\sum_{k\leqq n} A_{n-k}^r a_k\varphi_k(x)=o(1)(\log n)^{\frac{1}{2}}$ für $r\geqq\frac{1}{2}$, und $\sigma_n^{(r)}(x)=o(1)(\log n)^{\frac{1}{2}+\varepsilon}$ für $0<r<\frac{1}{2}$ und jedes $\varepsilon>0$ gilt. Der Beweis stützt sich auf ein Lemma von Kaczmarz und Zygmund nach dem, für $r>\frac{1}{2}$, fast überall $\sum n[\sigma_n^{(r)}(x)-\sigma_{n-1}^{(r)}(x)]^2<\infty$ ist. Für dieses Lemma wird ein neuer Beweis gegeben. Gilt noch $\sum a_n^2\log^{2-\varepsilon}n<\infty$, $0<\varepsilon\leqq 1$, so ist fast überall $\sigma_n^{(1)}(x)=f(x)+o(1)(\log n)^{-\frac{1}{2}(1-\varepsilon)}$.

A. Peyerimhoff (Giessen).

Vzorova, A. I. On construction of polynomials which are orthogonal on a family of ellipses. Vyčisl. Mat. **1** (1957), 120–130. (Russian)

Construction de polynômes harmoniques

$$v_k = v_k(r, \varphi) = \sum_{l=0}^{k} c_{k,l} u_l,$$

combinaisons linéaires de $u_{2k-1} = r^k \cos k\varphi$ et $u_{2k} = r^k \sin k\varphi$, orthogonaux sur l'ellipse $x = a \cos E$, $y = b \sin E$, dans le sens que $\int_0^{2\pi} v_k v_l dE = \delta_k^l$. Calcul explicite des $c_{k,l}$. Les résultats, quoique clairement exprimés, sont trop compliqués pour être reproduits. Application à la résolution approchée du problème de Dirichlet dans l'ellipse.

J. P. Kahane (Montpellier).

Paszkowski, S. On approximation with nodes. Rozprawy Mat. **14** (1957), 63 pp.

Let $T = \{t_1, \cdots, t_m\}$ be a fixed set of points of $[a, b]$. For a continuous function $\xi(t)$, $a \leq t \leq b$, $\varepsilon_n(\xi)$ denotes the degree of uniform approximation of ξ by polynomials ω_n of degree not exceeding n; while $\varepsilon_n(\xi, T) = \min \max_t |\xi(t) - \omega_n(t)|$ (where the minimum is taken for all ω_n satisfying the conditions $\omega_n(t_i) = \xi(t_i)$, $i = 1, \cdots, n$) is the degree of approximation of ξ with nodes t_i. In the first part of the paper the author proves theorems of the type (*) $\varepsilon_n(\xi, T) \leq C_n \varepsilon_n(\xi)$ for $n \geq N$, where N and C_n depend on T and a, b but not on ξ. One can take $C_n = 2$ with some unspecified N [for a somewhat better result see Paszkowski, Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 745–748; MR **19**, 137]. Some other values for C_n and N are also given; the case $m = 2$ is discussed in detail. The second part deals with properties of two best approximations ω_n, ω_{n+1} of degrees $\leq n$ and $n+1$. Properties of the roots of the equation $\omega_n(x) = \omega_{n+1}(x)$ and of the Tchebyshev points of ω_n and ω_{n+1} are considered. This leads to variants of the known theorem which describes the rate of growth of a polynomial of degree n outside of $[-1, 1]$ in terms of the growth of the Tchebyshev polynomial T_n. Suppose that T, two polynomials ω_n, ω_{n+1} of degrees $\leq n$ and $n+1$ respectively, and numbers $0 \leq e_{n+1} < e_n$ are given. Rather involved necessary and sufficient conditions are derived for the existence of a continuous function ξ for which ω_n and ω_{n+1} are polynomials of best approximation with nodes T and with $\varepsilon_i(\xi, T) = e_i$, $i = n, n+1$.

G. G. Lorentz (Syracuse, N.Y.).

Paškovskiĭ, S. F. [Paszkowski, S.] The position of the (e)-points of polynomials of best approximation. Dokl. Akad. Nauk SSSR (N.S.) **117** (1957), 576–577. (Russian)

Let $E_n(f)$ be the deviation of $f(x)$ on $[-1, 1]$ from the polynomial $p_n(x)$ of best uniform approximation. The author states that if $E_{n+1}(f)/E_n(f) \leq g < 1$, then the intervals $I_k = [t_{n,k,h}, -t_{n,n+1-k,h}]$, $k = 0, 1, \cdots, (n+1)$, contain points u_k such that $f(u_k) - p_n(u_k) = \pm E_n(f)$ with alternating signs. Here $h = (1-g)/(1+g)$ and $t_{n,k,h}$ is the largest root not exceeding v_k of the equation $|T_{n,k,h}(t)| = h$; the polynomial $T_{n,k,h}$ of degree $(n+1)$ and the numbers v_i are defined by $T_{n,k,h}(v_i) = (-1)^{n+1-i} h$, $i = 0, \cdots, k-1$, $T_{n,k,h}(v_i) = (-1)^{n+1-i}$, $i = k, \cdots, n+1$, $T'_{n,k,h}(v_i) = 0$, $i = 1, \cdots, n$, $-1 = v_0 < v_1 < \cdots < v_{n+1} = 1$. This can be used for the computation of p_n if the intervals I_k are disjoint.

G. G. Lorentz.

Ganzburg, I. M. A generalization of some results of S. M. Nikolskiĭ and A. F. Timan. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 727–730. (Russian)

The class $MH^{(\omega)}$ consists of functions f on $[-1, 1]$ such that $|f(x') - f(x'')| \leq M|x' - x''|^{\omega}$. Let $s_n(f, x)$ be the partial sums of the expansion of f in a series of Chebyshev polynomials and let $\sigma_{n,k}(f, x) = (k+1)^{-1} \sum_{\nu=n-k}^{n} s_\nu(f, x)$. Let $E_n(x)$ be the supremum of $|f(x) - \sigma_{n,k}(f, x)|$ for $f \in MH^{(\omega)}$. The asymptotic value of $E_n(x)$ was calculated for $k = 0$, $\alpha = 1$ by Nikolskiĭ [Izv. Akad. Nauk SSSR. Ser. Mat. **10** (1946), 295–322; MR **8**, 153]; and for $k = 0$, $0 < \alpha < 1$, by Timan [same Dokl. **77** (1951), 969–972; MR **12**, 823]. The author finds

$$E_n(x) = \frac{2^{\alpha+1} M}{\pi^2 n^\alpha} (1 - x^2)^{\frac{1}{2}\alpha} \int_0^{\frac{1}{2}\pi} t^\alpha \sin t dt \cdot \log \frac{n+1}{k+1} + O(n^{-\alpha})$$

where the O-term is uniform in x and k for $0 < \alpha < 1$, but only for $0 \leq k \leq n\theta$, $\theta < 1$, if $\alpha = 1$.

R. P. Boas, Jr.

Timan, A. F. Converse theorems in the constructive theory of functions given on a finite segment of the real axis. Dokl. Akad. Nauk SSSR (N.S.) **116** (1957), 762–765. (Russian)

The author proves the following theorem. Let f be a function on $(-1, 1)$ with modulus of continuity $\omega(f, h)$. If there is a sequence of polynomials $P_n(x)$ such that

$$|f(x) - P_n(x)| \leq \omega\{n^{-1}[(1 - x^2)^{\frac{1}{2}} + n^{-1}|x|]\},$$

then there is a constant C such that

$$\omega(f, h) \leq Ch \int_h^1 u^{-2} \omega(u) du, \quad 0 < h \leq \tfrac{1}{2}.$$

He gives a similar theorem in which the conclusion is a bound for $\omega(f^{(r)}, h)$.

R. P. Boas, Jr. (Evanston, Ill.).

Luke, Yudell L. Rational approximations to the exponential function. J. Assoc. Comput. Mach. **4** (1957), 24–29.

If $R_n(u)$ is a polynomial of degree n, successive integrations by parts lead to $I_n \equiv z^{n+1} \int_0^1 e^{-u} R_n(u) du = H_n(z) - e^{-z} G_n(z)$, with $H_n(z) = \sum_{k=0}^{n} z^k D^{n-k} R_n(0)$, $G_n(z) = \sum_{k=0}^{n} z^k D^{n-k} R_n(1)$, whence $e^{-z} = H_n(z)/G_n(z) - r_n$, $r_n = I_n/G_n$. Here H_n/G_n gives a rational approximation to e^{-z}, while r_n represents the corresponding error. The polynomial $R_n(u)$ may be selected so as to minimize the error in some sense. Selecting Jacobi polynomials (normalized for the interval (0, 1)) as his $R_n(u)$, one finds that for the best approximation in the sense of Tchebyshev, G_n and H_n turn out to be essentially Bessel polynomials {Remark: long before the reviewer mentioned the approximation of the exponential function by Bessel polynomials [§ 2 in Trans. Amer. Math. Sci. **71** (1951), 197–210; MR **14**, 747], this property was used by Hermite in his proof of the trancendency of e [Oeuvres, Gauthier-Villars, Paris, 1905–17, v. 3, p. 153].} Under conditions that are not completely clear to the reviewer, it is found that $R_n(u)$ is either Tchebyshev's polynomial T_n, or the nth Legendre polynomial (both normalized for (0, 1)). The corresponding approximations of the exponential coincide with those given by the main diagonal of the Padé table. The foregoing results are used to approximate the exponentials occurring in the characteristic equation, obtained when Laplace transforms are applied to some linear differential equations. A concrete example is worked out numerically.

E. Grosswald.

Trigonometric Series and Integrals

Chen, Yung-Ming. Some asymptotic properties of Fourier constants and integrability theorems. Math. Z. **68** (1957), 227–244.

The author is first concerned with the cosine series $f(x)=\frac{1}{2}\lambda_0+\sum_1^\infty \lambda_n \cos nx$ or sine series $g(x)=\sum_1^\infty \lambda_n \sin nx$ with $\lambda_n \downarrow 0$. He shows that if $\beta>1$ and $1-\beta<\gamma<0$, then $x^{-\gamma}f(x)^p \in L$ if and only if $\sum n^{\gamma+p-2}\lambda_n{}^p<\infty$, and that the same is true for g. Now let $\Phi(x)$ be nonnegative and increasing, with $\Phi(x)/x$ increasing, while $\Phi(x)/x^k$ decreases for some $k>1$. Then, if $0<\gamma<1$, $x^{-\gamma}\Phi(|f(x)|) \in L$ if, and $x^{-\gamma}\Phi(|g(x)|) \in L$ if and only if, $\sum n^{\gamma-2}\Phi(n\lambda_n)$ converges. Similar results are given for Laplace transforms. Let $e^{-xt}\vartheta(t) \in L(0, \infty)$, with $\vartheta(t)>0$ and nonincreasing, and $\omega(x)$ the Laplace transform of $\vartheta(t)$. Then if $-\infty<\gamma<1$, $x^{-\gamma}\Phi(\omega(x)) \in L(0, 1)$ if and only if $t^{\gamma-2}\Phi((t\vartheta(t)) \in L(1, \infty)$; and the intervals $(0, 1)$ and $(1, \infty)$ can be interchanged. For the power series $F(x)=\sum c_n x^n$ with decreasing coefficients, the author proves that $(1-x)^{-\gamma}\Phi(|F(x)|) \in L(0, 1)$ if and only if $\sum n^{\gamma-2}\Phi(nc_n)<\infty$. He discusses special cases and gives some further results.

R. P. Boas, Jr.

Tolstov, G. P. On convergence of trigonometric Fourier series for continuous functions. Mat. Sb. N.S. **44(86)** (1958), 549–550. (Russian)

The author says that a function is absolutely continuous in the generalized sense in $[a, b]$ if this interval is the union of countably many closed sets on each of which it is absolutely continuous. Men'šov proved [same Sb. **11 (53)** (1942), 67–96; MR **7**, 59] that, if f is continuous and of period 2π, then for each $\varepsilon>0$ there are a set P of measure greater than $2\pi-\varepsilon$ and a function g such that $f(x)=g(x)$ on P and the Fourier series of g converges uniformly on $[-\pi, \pi]$. Using this result, the author shows that if f is continuous its Fourier series is equiconvergent on sets of measure arbitrarily near 2π with the Fourier series of a function which is absolutely continuous in the generalized sense. Hence if one could show that the Fourier series of a function which is absolutely continuous in the generalized sense converges almost everywhere, the same would be true for all continuous functions.

R. P. Boas, Jr.

Manaresi, Fabio. Alcuni teoremi sulle serie coniugate della serie di Fourier di una funzione di più variabili. Rend. Sem. Mat. Univ. Padova **27** (1957), 181–192.

Es handelt sich um eine Verallgemeinerung einiger Ergebnisse von O. Szàsz [Duke Math. J. **4** (1938), 401–407] auf mehrfache Fourierreihen. Unter anderem beweist der Verf.: Ist $f(x, y)$ integrierbar in $0\leqq x, y\leqq 2\pi$ und periodisch mit Periode 2π in x und y, ist ferner $\int_0^\pi |f(x, y)|dx\leqq L$, $\int_0^\pi |f(x, y)|dy\leqq L$, so folgt aus der Existenz eines $\varphi(x, y)$ mit

$$\int_0^\pi\int_0^\pi |f(x+u, y+v)+f(x+u, y-v)$$

$$-f(x-u, y+v)-f(x-u, y-v)-\varphi(x, y)|dudv=o(uv)$$

$(u, v\to+0)$ die Beziehung $\sigma_{2m,n}-\sigma_{m,n}\to(\log 2)(2\pi)^{-1}\varphi(x, y)$ $(m,\ n\to\infty)$ für die $C_{1,1}$-Mittel σ_{mn} der konjugierten Fourierreihe bezüglich x oder y von $f(x, y)$.

A. Peyerimhoff (Giessen).

See also: Functions of Real Variables: Hobson.

Integral Transforms

Wintner, Aurel. Stratifications of Cauchy's "stable" transcendents and of Mittag-Leffler's entire functions. Amer. J. Math. **80** (1958), 111–124.

For γ in $0<\gamma<1$ there is a function $G(u, \gamma)$ in $0<u<\infty$ with $d_uG(u, \gamma)\geqq 0$ such that

$$\exp(-t^\gamma)=\int_0^\infty e^{-tu}d_uG(u, \gamma).$$

Starting from this known fact, the author observes that for $0<\alpha<\beta<\infty$, $0<t$, $u<\infty$, on putting $\gamma=\alpha/\beta$, $F(u, t, \alpha, \beta)=G(ut^{-1/\gamma}, \alpha/\beta)$ one has, for all $\lambda>0$,

$$\exp(-t\lambda^\alpha)=\int_0^\infty \exp(-u\lambda^\beta)d_uF(u, t, \alpha, \beta)$$

with $d_uF(u, t, \alpha, \beta)\geqq 0$.

Now, for a sequence $0=\lambda_0<\lambda_1<\lambda_2<\cdots$, and any functions $\varphi_n(x)$ of any variable x, for the functions $\phi_\alpha(t, x)=\sum_n \exp(-t\lambda_n{}^\alpha)\varphi_n(x)$, this implies

$$\phi_\alpha(t, x)=\int_0^\infty \phi_\beta(u, x)d_uF(u, t, \alpha, \beta).$$

{The author restricts himself, unnecessarily, to the functions $\phi_\alpha(t, x)=1+2\sum_{n=1}^\infty \exp(-tn^\alpha)\cos nx$, and to some similar special functions in which λ_n is a continuous parameter instead of being a discrete one as above.}

S. Bochner (Princeton, N.J.).

Stanković, Bogoljub. Abbildung gewisser Operationen durch die zweidimensionale Laplace-Transformation. Acad. Serbe Sci. Publ. Inst. Math. **11** (1957), 1–8.

Writing

$$\Phi(\mu, \nu; z)=\sum_{k=0}^\infty \frac{z^k}{\Gamma(k+1)\Gamma(\mu+\nu k)},$$

and

$$\phi(u, v)=\int_0^\infty\int_0^\infty e^{-ux-vy}f(x, y)dxdy\equiv\mathscr{L}[f(x, y)]$$

it is proved that

$$\mathscr{L}\Big[\int_0^\infty\int_0^\infty x^{\mu-1}A(\cdot)y^{\beta-1}B(\cdot)f(\xi, \eta)d\xi d\eta\Big]=u^{-\mu}v^{-\beta}\phi(u^{-\nu}, v^{-\rho}),$$

where

$$A(\cdot)\equiv\Phi(\mu, \nu; -\xi x^\nu),\ B(\cdot)\equiv\Phi(\beta, \rho; -\eta y^\rho).$$

The parameters, μ, ν are not necessarily integers, in fact $\mu>0$ and $\nu>-1$. The result is thus a generalisation of formulae, p. 186 (40) and p. 187 (51), of D. Voelker and G. Doetsch, Die zweidimensionale Laplace-Transformation [Birkhäuser, Basel, 1950; MR **12**, 699].

J. L. Griffith (Kensington).

Artémiadis, Nicolas K. Sur une sous-classe de la classe $L_1(-\infty, \infty)$. C. R. Acad. Sci. Paris **246** (1958), 1648–1650.

Let $f \in L(-\infty, \infty)$ and let F be its Fourier transform. The author applies an inequality which he gave earlier [same C. R. **240** (1955), 1500–1502; **244** (1957), 544–547; MR **16**, 817; **18**, 575] to show that if $f(x)\not\equiv 0$, $\Re F(\alpha)\geqq 0$ and there is a c such that $\int_0^1 t^{-1}|f(t)-c|dt<\infty$, then $\Re c>0$. He then defines the class $\mathscr{D}$ as consisting of elements g of L such that $\Re G(x)$ does not change sign, $\int_0^1 t^{-1}|g(t)-c|dt<\infty$ for some c, and $\|g(x+t)-g(x)\|_L=O(t^\varepsilon)$ $(\varepsilon>0,\ t\to 0)$. He shows that $\mathscr{D}$ is closed under multiplication and that c is necessarily real.

R. P. Boas, Jr. (Evanston, Ill.).

Berg, Lothar. Asymptotische Entwicklung einer Klasse von Integralen. Math. Nachr. **16** (1957), 207–214.

The main theorem was announced in Z. Angew. Math. Mech. **36** (1956), 245–246 [MR **18**, 207]; the author remarks that the condition $f^{(\nu+1)}(s)=O(\sigma^{\alpha-1}|f^{(\nu)}(\sigma)|)$ quoted in the review must hold uniformly in τ; on the other hand, it is only required that $f(s)$ is analytic in some half-strip $\sigma\geqq T$, $\tau_1\leqq\tau\leqq\tau_2$, rather than in a half-plane. The theorem is proved and some special cases are discussed.

R. P. Boas, Jr. (Evanston, Ill.).

Kumar, Ram. Some theorems connected with generalised Hankel-transform. Riv. Mat. Univ. Parma **7** (1956), 321–332.

The author proves two theorems about a generalization of the Hankel transformation due to Agarwal [Ann. Soc. Sci. Bruxelles. Sér. I. **64** (1950), 164–168; MR **12**, 605]. He applies these theorems to evaluate a number of definite integrals.

P. G. Rooney (Toronto, Ont.).

See also: Trigonometric Series and Integrals: Chen.

Ordinary Differential Equations

Strodt, Walter. Principal solutions of ordinary differential equations in the complex domain. Mem. Amer. Math. Soc. no. **26** (1957), 107 pp. $2.00.

In an earlier paper [same Mem. no. **13** (1954); MR **16**, 702] Strodt initiated a study of the behavior of solutions of minimal rate of growth for large $|x|$ of a non-linear differential equation of the form

$$P(x, y, y', \cdots, y^{(n)})=0, \tag{A}$$

where P is a polynomial in y, y', $\cdots$, $y^{(n)}$, with coefficients which are analytic in x and satisfy certain other conditions. This second paper on the same subject is another important step forward in the attack on this very involved and difficult problem.

The new results can be described here only in a rough fashion, because of the highly technical nature of the restrictions and vocabulary. In fact the mere list of new terms and symbols with page references given at the end of this paper covers more than two pages. This list and the 1954 paper are indispensable for a reader who wishes to follow the details.

In the first paper one-term approximate solutions of minimal rate of growth were found for a broad class of nth-order non-linear equations of type (A); but corresponding exact solutions were found only for certain first-order equations.

In the present paper the analysis of first-order equations is made more precise and efficient. Furthermore, exact solutions of minimal rate of growth are obtained for a broad class of non-linear equations of type (A) of arbitrary finite order n. The results are also extended to equations of infinite degree in y and its derivatives. Two examples illustrating the use of the theory are discussed.

The chief devices utilized are: (1) the algorithm of the principal monomial (this is Stodt's powerful extension of the Newton polygon construction to obtain one-term logarithmic monomial approximations to solutions); (2) precise analysis of a first-order non-homogeneous equation; (3) successive approximations; (4) attenuation of the quadratic and higher-degree terms by means of transformations $y(x)=y_0(x)+z(x)$, where $y_0(x)$ is a solution of a differential equation approximating the given equation; plus a new and effective device, namely (5) approximate factorization of an algebraic differential operator into first-order homogeneous linear operators.

Two types of problems are considered, the so-called uniformly quasi-linear and the asymptotically quasi-linear equations. In the uniformly quasi-linear case a count of the number of parameters which appear in the set of solutions vanishing "smoothly" at infinity in an appropriate narrow sector is obtained. Analytic continuation of solutions into adjacent sectors is considered. The asymptotically quasi-linear equations are treated by reducing them to the uniformly quasi-linear case; but in carrying out this reduction the parameter count becomes obscure and will require further study.

H. L. Turrittin (Minneapolis, Minn.).

Mikołajska, Z. Sur l'instabilité d'une solution périodique isolée d'une équation différentielle dépendant d'un paramètre. Ann. Polon. Math. **4** (1958), 127–136.

Consider the real scalar equation $\dot{x}=f(t, x)$, f continuous, periodic in t with period $T>0$; conditions for the uniqueness of solutions are assumed to be satisfied. Let $\varphi_0(t)$ be an isolated periodic solution of period T. Theorem A: For the instability of φ_0 (i.e., φ_0 neither positively nor negatively stable) it is sufficient that there exist a continuous function $F(t, x, \lambda)$, $F(t, x, 0)=f(t, x)$, F periodic in t with period T, such that there are numbers L, $\rho>0$ such that $\dot{x}=F(t, x, \lambda)$ has exactly two periodic solutions of period T, $\psi_1(t, \lambda)$, $\psi_2(t, \lambda)$, for $0<\lambda\leqq L$, satisfying $|\psi_i(0, \lambda)-\varphi_0(0)|\leqq\rho$, $\lim_{\lambda\to 0}\psi_i(t, \lambda)=\varphi_0(t)$ and ψ_1 positively stable, ψ_2 negatively stable. Theorem B: If moreover f has continuous partials f_x, f_{xx} such that

$$\int_0^T f_{xx}(t, \varphi_0(t))\exp\left(\int_0^t f_x(s, \varphi_0(s))ds\right)dt<0,$$

the above condition is also necessary.

J. L. Massera (Montevideo).

Lykova, O. B. On one-frequency oscillations in systems with many degrees of freedom. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 222–226. (Ukrainian. Russian and English summaries)

The author considers a system of differential equations. It is assumed that the mechanical system, whose motion is described by equations of the form $dz/dt=Pz+\varepsilon F(t, z)$, where $P=\|p_{ik}\|$ is a matrix of order $(n+2)$ and $F(t, z)$ is an analytic function of z, periodic in t with period 2π, is performing a one-frequency oscillatory process. The characteristic equation has a pair of purely imaginary roots: $\pm\lambda i$, while the other roots have negative real parts. By these relationships an approximate particular solution is found, corresponding to the one-frequency regime.

Author's summary.

Zoobowa, A. F. Concerning oscillation of the solutions of an equation of the second order. Vestnik Leningrad. Univ. **12** (1957), no. 1, 168–174, 211. (Russian. English summary)

The author presents a number of results concerning the non-oscillation of solutions of $(r(x)u')'+p(x)u=0$, where $r(x)>0$. A typical result is the following. A necessary and sufficient condition for non-oscillation is the existence of a function $g(x)$, non-vanishing for $x\geqq b$, with the property that for $x\geqq b$ one has the inequality

$$0<[c-\int_b^x g(x)p(x)dx]/g'(x)\leqq r(x).$$

R. Bellman (Santa Monica, Calif.).

Gutowski, Roman. Free vibration of a system of one degree of freedom with non-linear elastic characteristic, taking into consideration linear viscous damping. Arch. Mech. Stos. **9** (1957), 647–668. (Polish and Russian summaries)

The purpose of the present paper is to find sufficient conditions for the solution of the equation (1) $x''+2hx'+\alpha^2\varphi(x)=0$, $x(0)=a$, $x'(0)=b$, $a^2+b^2>0$, $0\leq t<+\infty$, to be oscillatory; that is, that the solution $x(t)$ have an infinite number of zeros in $[0, +\infty)$. The main theorem is the following: If $\varphi(x)$ is strictly monotone increasing, analytic and entire in $(-\infty, +\infty)$, $\varphi(-x)=-\varphi(x)$, $\varphi(0)=0$, $\lim_{x\to\pm\infty}\varphi(x)=\pm\infty$, then the solution of (1) is oscillatory for arbitrarily large deflections x provided that the line L: $y=(h/\alpha)^2x$, $(h/\alpha)^2<1$, is such that $\varphi(x)$ is above L for $x>0$ and below L for $x<0$. If $\varphi(x)$ intersects L at a point other than $x=0$, then an oscillatory solution exists only for sufficiently small deflections. Both $x(t)$ and $x'(t)$ of (1) have an infinite number of zeros; the zeros of $x'(t)$ separate those of $x(t)$ and both functions oscillate around the line $x=0$. A lower bound is given for the distance between consecutive zeros of $x(t)$ and the solution of (1) is discussed in the (x, x') plane. Also, a step by step integration procedure is given for finding the approximate solution $x(t)$ by assuming that $x''(t)$ varies linearly in t in a single step. The approximate integration procedure can be applied to equations more general than (1).

J. K. Hale (Baltimore, Mcl.).

Gambill, Robert A. A fundamental system of real solutions for linear differential systems with periodic coefficients. Riv. Mat. Univ. Parma **7** (1956), 311–319.

Consider the system of linear differential equations

$$(*)\qquad y_j''+\sigma_j^2y_j+\lambda\sum_{h=1}^{n}\varphi_{jh}(t)y_h=0\ (j=1, 2, \cdots, n),$$

where λ is a real parameter, $\sigma_1, \cdots, \sigma_n$ are positive numbers, $m\omega\neq 2\sigma_j$, $m\omega\neq\sigma_j\pm\sigma_h$, $j\neq h$, $j, h=1, 2, \cdots, n$, $m=0, 1, 2, \cdots$, and each $\varphi_{jh}(t)=\varphi_{jh}(t+2\pi/\omega)$, $\omega>0$. L. Cesari [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **11** (1941), 633–695; MR **8**, 208] gave a method of successive approximations, convergent for λ sufficiently small, for the determination of the characteristic exponents and a fundamental system of solutions of (*) under the conditions that the functions $\varphi_{jh}(t)$ have absolutely convergent Fourier series. This method was also shown to be convergent for the case where the functions $\varphi_{jh}(t)$ are L-integrable in $[0, 2\pi/\omega]$ [Hale, Riv. Mat. Univ. Parma **5** (1954), 137–167; MR **17**, 36]. Using this method, very general theorems for the boundedness of the solutions of (*) have been given. [For a bibliography of the application of this method to both linear and nonlinear differential equations, see MR **19**, 276.] Using this method, the author [ibid. **6** (1955), 37–43; MR **17**, 849] has previously given explicit expressions for the characteristic exponents of (*) in terms of the Fourier coefficients of the functions $\varphi_{jh}(t)$. In the present paper, the author gives explicit expressions (up through terms in λ^3) for a fundamental system of real solutions of (*) in terms of the Fourier coefficients of the $\varphi_{jh}(t)$ when it is known that all of the solutions of (*) are bounded. These expressions yield the well-known Mathieu functions when (*) is the Mathieu equation. {Reviewer's remark: in the same way, one may also find the explicit expression for the solutions when it is known that some solutions are unbounded.} *J. K. Hale* (Baltimore, Md.).

Hale, Jack K. Sufficient conditions for the existence of periodic solutions of systems of weakly nonlinear first and second order differential equations. J. Math. Mech. **7** (1958), 163–171.

The author, using the method which he, L. Cesari and R. A. Gambill have developed [see Hale, Illinois J. Math. **1** (1957), 98–104; MR **19**, 276], studies systems of the form

$$y''+Dy=\varepsilon f(y, w, y', \varepsilon, t),\ w'=\varepsilon g(y, w, y', \varepsilon, t),$$

where y and f are μ-dimensional real vectors, w and g are $(n-\mu)$-dimensional real vectors, ε is real, $D=\operatorname{diag}(\sigma_1^2, \cdots, \sigma_\mu^2)$ with $\sigma_i>0$, $i=1, \cdots, \mu$. It is supposed that f and g have convergent power series expansions in ε and the components of y, w, y' in some neighborhood of $(0, \cdots, 0)$ independent of t, the expansion coefficients having the period T in t. The general results are of the same type as in earlier papers by Hale and others and are applied to the (scalar) equation (*) $y'''+\sigma^2y'=\varepsilon f(y, y', y'', \varepsilon)$ to obtain the following: If $f(y, -y', y'', \varepsilon)=-f(y, y', y'', \varepsilon)$ and y_0, y_0', y_0'' are any numbers such that $|y_0|$, $|y_0'|$, $|y_0''|<C$, where C is a constant, then there exists, for ε sufficiently small, a periodic solution $y(\varepsilon, t)$ of (*) such that $y(\varepsilon, 0)=y_0$, $y'(\varepsilon, 0)=y_0'$, $y''(\varepsilon, 0)=y_0''$. Furthermore, every point of the form $(y, 0, 0)$ in the (y, y', y'') phase space is stable in the sense of Liapounoff.

C. E. Langenhop (Ames, Iowa).

Minorsky, N. Structure topologique de l'équation de M. Liénard. J. Phys. Radium (8) **18** (1957), supplément, 121A–130A.

Using the "first variation", the author studies bifurcation phenomena of equations of the form $\ddot{x}+f(x, \dot{x})\dot{x}+x=0$, where f is a polynomial with small coefficients. Polar coordinates in the $(x, \dot{x})$ plane are used as the new dependent variables. *N. Levinson.*

Tschobanow, Iw. Über eine Differentialgleichung von T. Leko. Jber. Deutsch. Math. Verein. **60** (1958), Abt. 1, 115–116.

Der Verfasser zeigt, dass die Differentialgleichung (1) $yy''+[q(x)y]'y=F[y'+q(x)y]$ durch Quadraturen integrierbar ist. *M. Zlámal* (Brno).

Tschobanow, Iw.; und Paskalew, G. Über eine Klasse von Differentialgleichungen. Jber. Deutsch. Math. Verein. **60** (1958), Abt. 1, 116–118.

Es wird gezeigt, dass eine Klasse von Differentialgleichungen welche die Differentialgleichung (1) aus der vorangehenden Rezension als Spezialfall enthält, durch Quadraturen lösbar ist. *M. Zlámal* (Brno).

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Foiaş, Gussi, et Poenaru. **Classical Thermodynamics, Heat Transfer:** Reid.

Partial Differential Equations

Mikusiński, Jan. Problèmes initiaux et mixtes pour les équations aux dérivées partielles considérés du point de vue du calcul opérationnel. Czechoslovak Math. J. **6(81)** (1956), 311–317. (Russian summary)

Expository lecture on the author's operational calculus on a finite interval [Studia Math. **15** (1956), 225–251; MR **18**, 585] and its application to linear partial differential equations with constant coefficients. *A. Erdélyi.*

Finn, Robert; and Gilbarg, David. Three-dimensionalsubsonic flows, and asymptotic estimates for elliptic partial differential equations. Acta Math. 98 (1957), 265–296.

This interesting paper is concerned with questions of existence, uniqueness and asymptotic behaviour for non-viscous compressible fluid flows past a bounded obstacle in 3-dimensional space, for which the velocity at infinity is subsonic. In pure mathematical terms the concern is with those solutions, of a quasi-linear equation

$$(1) \qquad a_{ij}(u_k)\frac{\partial^2\phi}{\partial x_i \partial x_j}=0, \quad u_k=\frac{\partial\phi}{\partial x_k}, \quad a_{ij}=a_{ji},$$

in $n\geqq 3$ variables x_i, for which (i) each u_k tends to a prescribed limit at infinity, such that near infinity the equation is elliptic, and (ii) $\partial\phi/\partial\nu=0$ on an interior boundary B.

The vital step (§ 4) is to prove that for each u_k the Dirichlet integral $\int(\nabla u_k)^2 dV$, over the region between spheres of radii r, R, converges as $R\to\infty$, and to estimate its rate of convergence to zero as $r\to\infty$; this is achieved by using transformations between volume and surface integrals and inequalities of Schwarz and Wirtinger. An estimate of the rate of convergence of u_k then follows (§§ 3, 5). By an adaptation of the classical argument used for Laplace's equation the uniqueness theorem now follows, for flows which are everywhere subsonic outside the obstacle B but may possibly reach sonic speed on B; if such a flow, with prescribed velocity at infinity, exists it is proved that there is no other smooth flow, subsonic or even mixed, with this velocity at infinity; and the force on B is proved to vanish (d'Alembert's paradox).

Finally, with the help of the fixed-point theorem of Leray and Schauder, an existence theorem is proved for flow past an obstacle for which the maximum speed q_m is sufficiently within the subsonic range; for $\gamma=1.5$, the restriction that the Mach number be less than 0.53 is sufficient.

In an addendum, a theorem of Payne and Weinberger [see next review] is applied to get an improved estimate of the rate of convergence of the u_k to their limits.

T. M. Cherry (Melbourne).

Payne, L. E.; and Weinberger, H. F. Note on a lemma of Finn and Gilbarg. Acta Math. 98 (1957), 297–299.

This paper has the same context as that of Finn and Gilbarg, above. It is shown that if u satisfies $(a^{ij}u_{,i})_{,j}=0$, and is such that a^{ij} (which depends on u) approaches the unit matrix of order n at infinity, then $\int a^{ij}u_{,i}u_{,j}dV$, over the region outside a sphere of radius r, is for large r of order $r^{2-n+o(1)}$. This is essentially a sharp result because for Laplace's equation the order is r^{2-n}. *T. M. Cherry.*

Davis, Robert B. A reduction-of-order theorem. J. Math. Phys. 36 (1957), 164–166.

The author derives an asymptotic formula, as $\varepsilon\to 0$, for the solution of the boundary value problem $\varepsilon^2\Delta u_\varepsilon+a(x,y)u_\varepsilon=f(x,y)$ in R, $u_\varepsilon=b(x,y)$ in B, where R is a bounded region with boundary B. It is assumed that a, b, f and B are in C^6 and that $a(x,y)>0$. A straight-forward application of a method of Levinson [Ann. of Math. (2) 51 (1950), 428–445; MR 11, 439] yields the result that $u_\varepsilon=fa^{-1}+e^{-g/\varepsilon}h+O(\varepsilon^2)$ in $R+B$, where $g=g(x,y)$ and $h=h(x,y)$ are solutions of certain first order partial differential equation and $g>0$ in R. {The author's statement that this problem had been previously considered in a paper by the reviewer is incorrect.}

W. Wasow (Madison, Wis.).

Lieberstein, H. M. On the generalized radiation problem of A. Weinstein. Pacific J. Math. 7 (1957), 1623–1640.

The generalized radiation problem requires the determination of a non-singular solution of the two-dimensional Euler-Poisson-Darboux equation ("EPD equation")

$$(*) \qquad u_{xx}^{[k]}=u_{yy}^{[k]}+\frac{k}{y}u_y^{[k]}$$

for $-\infty<k<1$ such that

$(**)$ $\lim_{y\to 0} u^{[k]}(x,y)=f(x)$ and $u^{[k]}(x,y)=0$ for $y=x$.

$f(x)$ is a function given on some interval $0\leqq x\leqq a$, possessing a specified number of continuous derivatives there and having another specified number of zero derivatives at $x=0$. If k is an integer and $u^{[k]}$ a solution of the above generalized radiation problem, then $u^{[2k]}(x,y)=u^{[k]}(x,y)/y^{1-k}$ is a solution of the classical radiation problem in a $(3-k)$-dimensional space. The author deals with uniqueness for the generalized radiation problem. He considers solutions which have two continuous derivatives on $y=x$. A function is called regular in the triangle T with vertices $(0,0)$, $(a,0)$, $(a/2,a/2)$ if it has two continuous derivatives in some triangle G whose interior contains T and its sides except for the base line $y=0$. Only a function satisfying the EPD equation, regular in T, and taking on the given data will be considered a solution of the radiation problem. The author proves that $u^{[k]}(x,0)=u^{[k]}(x,x)\equiv 0$ implies $u^{[k]}(x,y)\equiv 0$. In his considerations the recursion $u_y^{[k]}(x,y)=yu^{[k+2]}(x,y)$ plays a basic role (in their n-dimensional form these recursions are due to A. Weinstein). Any uniqueness proof which applies to solutions of (*), (**) also applies when it is required that $u^{[k]}(x,x)=g(x)$, $u^{[k]}(x,0)=f(x)$, where $f(x)$ and $g(x)\not\equiv 0$ are given functions. From the Weinstein solution, it can be seen that the region of determination of $f(x)$ defined for $0\leqq x\leqq a$ is the infinite strip bounded by the lines $y=x$ and $y=x-a$. The uniqueness question, however, can be restricted to consideration of the characteristic triangle T (one considers only the problem $f(x)=0$). The characteristic problem for the two-dimensional EPD equation was solved by Riemann to obtain the "Riemann function" for the EPD equation.

M. Pinl (Cologne).

Krzyżański, M. Évaluations des solutions de l'équation aux dérivées partielles du type parabolique, déterminées dans un domaine non borné. Ann. Polon. Math. 4 (1957), 93–97.

Consider the parabolic equation

$$(1) \qquad \sum_1^m a_{ij}U_{x_ix_j}+\sum_1^m b_kU_{x_k}+CU-U_t=f,$$

where the coefficients a_{ij}, b_k are continuous bounded functions of $X=(x_1, x_2, \cdots, x_n)$ and t throughout an unbounded domain D. Let $c(X,t)\leqq 0$, $f(X,t)\geqq 0$ and let f be continuous and of exponential order $e^{Kr^2}(r=\|X\|)$ in D. The form $\sum a_{ij}\lambda_i\lambda_j$ is assumed to be positive definite. The frontier of D is composed of the unbounded domains S_0 and S_T lying in the hyperplanes $t=0$ and $t=T$, respectively, and a surface σ which is nowhere tangent to any characteristic $t=$const of (1). Let $\Sigma=S_0+\sigma$. Then under an additional hypothesis which essentially assures the existence of a regular solution of the homogeneous part of (1) and the assumptions that the solution $U(X,t)$ of (1) is of exponential order e^{Kr^2} and $|U(x,t)|\leqq M$ on Σ it is shown that $|U(X,t)|\leqq M$ throughout D.

Two corollaries give modified bounds for $U(X, t)$ in D under certain additional conditions. *C. G. Maple.*

Peetre, Jaak. A generalization of Courant's nodal domain theorem. Math. Scand. **5** (1957), 15–20.

Let Ω be a compact subdomain of a Riemannian manifold M homeomorphic to a disc in the Euclidean plane. Let Γ denote the boundary of Ω, Ω_0 the smallest simply connected domain containing Ω, V_0 the area of Ω_0. The eigenvalue problem $\Delta u-\lambda u=0$ in Ω, $u=0$ on Γ (Δ=Beltrami-Laplace operator) is considered. Let $N(n)$ designate the number of nodal domains of the nth eigenfunction. The author generalizes results due to A. Pleijel [Comm. Pure Appl. Math. **9** (1956), 543–550; MR **18**, 315]. He proves, e.g., the inequality

$$\limsup_{n\to\infty} N(n)/n\leqq(2/j)^2<1$$

(j=first positive zero of the Bessel function J_0) for the case where either one of the following two additional assumptions is fulfilled: (I) $V_0B\leqq\pi$ (B=sup of the Gaussian curvature in Ω_0); (II) the Gaussian curvature is constant on M. *A. Huber* (Münchenstein).

Schwarz, Binyamin. Bounds for the principal frequency of the non-homogeneous membrane and for the generalized Dirichlet integral. Pacific J. Math. **7** (1957), 1653–1676.

Isoperimetric inequalities are obtained for the lowest eigenvalue of the problem

$$\Delta u+\lambda p(x, y)u=0 \tag{1}$$

in a domain D with $u=0$ on its boundary C. The function p is positive and continuously differentiable.

The functions p^+ and p^- are circularly symmetric functions defined on the circle D^* of the same area as D in such a way that the areas where $p>z$, $p^+>z$, and $p^->z$ agree for each real z, while p^+ is non-decreasing and p^- non-increasing in the distance r from the center of D^*.

It is shown that if λ^- is the lowest eigenvalue of the problem

$$\Delta v+\lambda^-p^-v=0 \text{ in } D^* \tag{2}$$

with $v=0$ on the boundary C^* for D^*, then (3) $\lambda\geqq\lambda^-$. This is a direct generalization of the isoperimetric inequality of Faber and Krahn.

It is noted that even if p is continuously differentiable, the function p^-, which is certainly continuous, may have discontinuous derivatives. In this case, the eigenvalue λ^- in the sense of the differential equation (2) may not exist. However, it is shown that λ^- can still be defined by the means of the generalized minimum principle.

$$\lambda^-=\inf_{\varphi=0 \text{ on } C^*}\frac{\iint_{D^*}|\text{grad }\varphi|^2dxdy}{\iint_{D^*}p^-\varphi^2dxdy}. \tag{4}$$

If D is a circle so that $D=D^*$, it is shown that (5) $\lambda\leqq\lambda^+$, where

$$\lambda^+=\inf_{\varphi=0 \text{ on } C^*}\frac{\iint_{D^*}|\text{grad }\varphi|^2dxdy}{\iint_{D^*}p^+\varphi^2dxdy}. \tag{6}$$

Analogous results are obtained for the generalized capacity of Szegö as well as for some one-dimensional problems. *H. F. Weinberger* (College Park, Md.).

See also: Banach Spaces, Banach Algebras, Hilbert Spaces: Foiaş, Gussi and Poenaru. **Differential Geometry:** Muracchini. **Classical Thermodynamics, Heat Transfer:** Jäckel; Reid. **Quantum Mechanics:** Adams. **Geophysics:** Bragard.

Difference Equations, Functional Equations

See: Control Systems: Brown.

Calculus of Variations

Dedecker, Paul. A property of differential forms in the calculus of variations. Pacific J. Math. **7** (1957), 1545–1549.

In E. Cartan's treatment of the calculus of variations by means of the calculus of forms, he introduced a particular form ω. If the variational problem involves the $\int L(t, q^i, \dot{q}^i)dt$, then ω is given by the expression.

$$\frac{\partial L}{\partial\dot{q}^i}\,dq^i-\left(\dot{q}^i\frac{\partial L}{\partial q^i}-L\right)dt.$$

There are however other forms which might, at first glance, serve as well as the uniquely chosen form ω. The author in this paper gives a careful justification for the use of this particular form ω rather than some other form. *L. Auslander* (Bloomington, Ind.).

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Weston, J. D. Mean values for continuous functions on a compact group. Portugal. Math. **15** (1956), 105–110 (1957).

Following standard procedures the existence and uniqueness of a mean, for continuous functions on a compact group, are proved. The values of the functions are allowed to be more general than usual; the value space is any space X for which one has a function, from n-tuples of points in X (n variable) to X, with certain abstract properties of a mean value. *W. Ambrose.*

Urbanik, K. On the limiting probability distribution on a compact topological group. Fund. Math. **44** (1957), 253–261.

A sequence of probability distributions μ_n on a compact group G is called normal if for each k there exist arbitrarily large N such that $1\leqq s\leqq k$ implies $\mu_s=\mu_{s+N}$. The author obtains simple necessary and sufficient conditions for weak* convergence of the infinite convolution-product of the measures in a normal sequence, and identifies the limiting distributions when they exist as Haar measures on compact subgroups of G. Since in particular a constant sequence is normal the results here include those of Kloss [Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 453–455; MR **18**, 680]. There is some overlap with methods and results of the reviewer [Proc. Amer. Math. Soc. **5** (1954) 923–929; MR **16**, 796]; in addition the author makes ingenious use of Peter-Weyl theory. *J. G. Wendel.*

Conner, P. E.; and Floyd, E. E. Orbit spaces of circle groups of transformations. Ann. of Math. (2) **67** (1958), 90–98.

Let G be a continuous group of transformations oper-

ating on a space X. A point x is called singular if $gx=x$ for some g in G different from the identity. If $gx=x$ for all g, x is called fixed. The isotropy subgroup G_x at a point x consists of all g's such that $gx=x$. In the present work G is the circle group; X is always assumed to be compact. Suppose that X is metric, finite dimensional, locally connected, and that all singular points are fixed. Then if X and the fixed point set F are locally connected in the sense of integral cohomology (colc), so is the orbit space X/G; if X is LC^1, so is X/G; if X and F have trivial cohomology groups, so does X/G; if X is simply connected, so is X/G. Suppose X is a manifold with or without boundary. Then there are only a finite number of distinct isotropy subgroups G_x, and F is colc; if X has trivial cohomology, so does F; X/G is an absolute neighborhood retract; it is an absolute retract if X is. In the last part of the paper, the authors employ spectral sequences to compute the cohomology of X/G assuming X to be an n-sphere.

P. A. Smith (New York, N.Y.).

Lie Groups and Algebras

★**Gel'fand, I. M.; and Graev, M. I. Analogue of the Plancherel formula for the classical groups.** American Mathematical Society Translations, Ser. 2, Vol. 9, pp. 123–154. American Mathematical Society, Providence R.I., 1958. iii+231 pp. $5.00.

Translated by E. Hewitt from Trudy Moskov. Mat. Obšč. **4** (1955), 375–404 [MR **17**, 173].

★**Naĭmark, M. A. On the description of all unitary representations of the complex classical groups. I, II.** American Mathematical Society Translations, Ser. 2, Vol. 9, pp. 155–215. American Mathematical Society, Providence, R.I., 1958, iii+231 pp. $5.00.

Translated by E. Hewitt from Mat. Sb. N.S. **35(77)**, 317–356 (1954); **37(79)**, 121–140 (1955) [MR **16**, 567; **17**, 61].

★**Naĭmark, M. A. A continuous analogue of Schur's lemma and its application to Plancherel's formula for complex classical groups.** American Mathematical Society Translations, Ser. 2, Vol. 9, pp. 217–231. American Mathematical Society, Providence, R.I., 1958. iii+231 pp. $5.00.

Translated by E. Hewitt from Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 3–16 [MR **17**, 875].

Adler, Alfred. Characteristic classes of homogeneous spaces. Trans. Amer. Math. Soc. **86** (1957), 348–365.

Let G be a compact Lie group and K a closed connected subgroup of G. With G/K as base space, there are two natural principal fibre bundles: (1) the bundle of oriented orthonormal frames of G/K, where the structural group is the proper orthogonal group $O^+(N)$, $N=\dim G/K$, and (2) the coset bundle, where the total space is G and the structural group is K. The author discusses the characteristic rings of these two bundles. The following theorems are proved. (I) If K is abelian, then the Pontrjagin classes over G/K are all zero. (II) If G/K is either a symmetric space or a simply-connected compact homogeneous complex manifold, then the characteristic classes of degree greater than $\dim G/K-(\text{rank } G-\text{rank } K)$ are all zero. (III) An invariant symmetric tensor over the Lie algebra of K is determined by its value on a maximal abelian subalgebra. *H. C. Wang* (Evanston, Ill.).

Palais, Richard S. Imbedding of compact, differentiable transformation groups in orthogonal representations. J. Math. Mech. **6** (1957), 673–678.

Let G be a compact Lie group. A triple $\langle G, M, \varphi\rangle$, where M is a differentiable manifold and $\langle g, m\rangle \to \varphi_g(m)$ is a smooth map of $G\times M\to M$ satisfying $\varphi_g\circ\varphi_h=\varphi_{gh}$, is called a (differentiable) transformation group. If N is any submanifold of M invariant under the action of G, then in an obvious way $\langle G, N, \varphi\rangle$ is a transformation group. In particular, if ψ is a linear representation of G on a finite dimensional vector space V and N is an invariant submanifold of V, then $\langle G, N, \psi\rangle$ is a transformation group. This paper is devoted to proving that, if $\langle G, M, \varphi\rangle$ is a transformation group and O is a relatively compact open submanifold of M invariant under G, then $\langle G, O, \varphi\rangle$ is isomorphic to such a subtransformation group $\langle G, N, \psi\rangle$ of a suitable linear transformation group $\langle G, V, \psi\rangle$. A similar result was obtained by Mostow [Ann. of Math. (2) **65** (1957), 432–446; MR **19**, 291].

A. M. Gleason (Cambridge, Mass.).

Mostow, G. D. Fully reducible subgroups of algebraic groups. Amer. J. Math. **78** (1956), 200–221.

The algebraic groups are those of Chevalley [Théorie des groupes de Lie, v. II, III, Hermann, Paris, 1951, 1955; MR **14**, 448; **16**, 901; see also A. Borel, Ann. of Math. (2) **64** (1956), 20–82; MR **19**, 1195] and the characteristic is zero. Full reducibility always refers to operation on a specific vector space. An algebraic group is fully reducible if and only if the same is true of its Lie algebra (considered as endomorphisms of the same vector space), or equivalently, if and only if the radical of the group is a torus; full reducibility is thus preserved under rational representations. Any algebraic group G is of the form $G=M\cdot N$ (semi-direct), where N is the unipotent part of the radical of G and M is any maximal fully reducible subgroup of G, and any two such subgroups M are conjugate; this is related to the Wedderburn decomposition of the enveloping algebra of G, and similar results hold for Lie algebras. A fully reducible group of automorphisms of a Lie (or associative, or Jordan) algebra leaves invariant a maximal semisimple subalgebra.

M. Rosenlicht (Evanston, Ill.).

Freudenthal, Hans. Zur vollständigen Reduzibilität der linearen Darstellungen halbeinfacher Gruppen und zum E. E. Levischen Satz. J. Madras Univ. Sect. B. **27** (1957), 225–236.

Let Γ be a semi-simple Lie algebra of characteristic zero. Then the Killing form (X, Y) of Γ is nondegenerate, and the Casimir operator for any non-trivial representation of Γ has nonzero trace. From these facts the author deduces elegantly the Weyl theorem that any representation of Γ is completely reducible and the Levi theorem that any Lie algebra Γ can be written as $\Gamma=\Delta+\Omega$ (direct), where Δ is the radical of Γ and Ω a subalgebra of Γ. An interesting generalization of the Casimir operator of Γ is the symmetric k-tensor F over Γ defined by $(F, P)=f(P)$, where P runs over all symmetric k-tensors,

$$(X_1\times\cdots\times X_k, Y_1\times\cdots\times Y_k)=\prod(X_i, Y_i),$$

and f is an "invariant" linear function on symmetric h-tensors. The "invariance" of F under Γ is proved and used in the proof. *R. Ree* (Vancouver, B.C.).

Dixmier, J. L'application exponentielle dans les groupes de Lie résolubles. Bull. Soc. Math. France **85** (1957), 113–121.

Let $\mathfrak{g}$ be a soluble Lie algebra of finite dimension over the complex numbers. Each non-zero root φ of $\mathfrak{g}$ and each integer l define a linear variety $\mathfrak{g}_{\varphi,l}$ by the equation $\varphi(a)=2\pi il$, the exceptional varieties of $\mathfrak{g}$. If G is the simply connected complex Lie group of $\mathfrak{g}$, then each $\mathfrak{g}_{\varphi,0}$ corresponds to a normal subgroup G_φ of G; the cosets of G_φ in G are the exceptional varieties of G. They correspond to those of $\mathfrak{g}$ under the exponential mapping $X\to\exp X$, taken mod N (where N is the normal subgroup of G corresponding to the maximal nilpotent ideal of $\mathfrak{g}$). Let $\mathfrak{s}$ (S) be the closure of the union of the exceptional varieties in $\mathfrak{g}$ (G), with $\mathfrak{r}$ (R) its complement, and denote the exponential mapping of $\mathfrak{g}$ into G by Ω. Then (i) $\Omega(\mathfrak{s})\subseteq S$; (ii) the restriction $\Omega|\mathfrak{r}$ is an isomorphism of $\mathfrak{r}$ onto R qua analytic manifolds; (iii) the mapping Ω is not a local monomorphism or a local epimorphism at any $a\in\mathfrak{s}$; i.e., for every neighbourhood V of $a\in\mathfrak{s}$, $\Omega|V$ is not 1-1, and for some neighbourhood W of $a\in\mathfrak{s}$, $\Omega(W)$ is not a neighbourhood of $\Omega(a)$. It follows that, for the type of group considered, the conditions (α) Ω is a monomorphism, (β) Ω is an epimorphism, (γ) G is nilpotent, are equivalent.

If $\mathfrak{g}$ is as before but real instead of complex, the roots of $\mathfrak{g}$ may be defined by means of the complexified form $\tilde{\mathfrak{g}}$, and the assertions (i)–(iii) then hold for real groups. The condition for $\mathfrak{s}$ to be empty now reads: Every root of $\mathfrak{g}$ has the form $\varphi'+i\varphi''$, where φ', φ'' are real and proportional. An equivalent condition is that $\mathfrak{g}$ admit a quotient with a subalgebra isomorphic to the 3-dimensional algebra with multiplication table $[x,y]=-z$, $[x,z]=y$, $[y,z]=0$. *P. M. Cohn* (Manchester).

See also: Linear Algebra: Kostant. **Measure, Integration:** Goetz.

Topological Vector Spaces

Banaschewski, B. On the Weierstrass-Stone approximation theorem. Fund. Math. **44** (1957), 249–252.

Let the statement "The Weierstrass-Stone approximation theorem holds for a space E" mean that any subring of the ring $\mathfrak{C}(E)$ of bounded continuous real functions on E that contains the constants and separates the points of E is uniformly dense in $\mathfrak{C}(E)$. It is known that when E is a completely regular Hausdorff space, the Weierstrass-Stone approximation theorem holds for E if and only if E is compact [Hewitt, Duke Math. J. **14** (1947), 419–427; MR **9**, 95]. In this note, the following generalization is obtained: If E is a space such that $\mathfrak{C}(E)$ separates the points of E, the Weierstrass-Stone approximation theorem holds for E if and only if every completely regular filter on E (in the sense of Bourbaki) has non-void intersection. The statement is proved by applying Hewitt's result to the completely regular space $\hat{E}$ associated with E for which $\mathfrak{C}(\hat{E})=\mathfrak{C}(E)$, to conclude that the approximation theorem holds for E if and only if $\hat{E}$ is compact, and then showing that this in turn is equivalent to the stated condition on E.

{Reviewer's remark: In the presentation of a class of examples of non-compact spaces X for which $\hat{X}$ is compact, the confusion of "countably compact" with "compact" leads to the erroneous statement that if E is locally compact and normal, then $\beta E-E$ cannot be finite. The ordinal space $W(\omega_1)$ is a counterexample. However, all that is desired is that $\beta E-E$ be non-discrete, which is true whenever E is not pseudo-compact.} *C. W. Kohls.*

Edwards, R. E. Note on two theorems about function algebras. Mathematika **4** (1957), 138–139.

Let X be a locally compact space, and $C(X)$ the algebra of continuous real (or complex) functions on X, endowed with the topology of compact convergence. If $f\in C(X)$, let Z_f denote the zero-set of f; and if $I\subset C(X)$, let $Z_I=\bigcap Z_f(f\in I)$. In this note concise proofs, based on integration theory as presented by Bourbaki, are given for the following two theorems. (1) If I is a closed ideal in $C(X)$, then I is precisely the set of f in $C(X)$ satisfying $Z_f\subset Z_I$. (2) If χ is a continuous homomorphism of $C(X)$ onto the scalar field, there is a point x of X such that $\chi(f)=f(x)$ for all $f\in C(X)$.

The author remarks that the arguments can also be applied to other function algebras, such as the algebra of continuous functions with compact supports (with the "strong" topology of Bourbaki), and that a similar proof using distribution theory yields the result that a closed ideal I in the space $C^p(X)$ of p-times continuously differentiable functions on a differentiable manifold X of class p contains each $f_0\in C^p(X)$ which, together with its partials of order $\leq p$, vanishes at each point of Z_I.

{The statement "$y\in X-S$" on p. 139, l. 9, should be "$y\neq x$".} *C. W. Kohls* (Urbana, Ill.).

Semadeni, Z.; and Zbijewski, P. Spaces of continuous functions. I. Studia Math. **16** (1957), 130–141.

The space $C_0(\Omega)$ of all continuous, real-valued functions on a σ-compact Hausdorff space Ω is characterized as a metric vector lattice. The characterization is obtained from that of Kakutani for compact Ω [Ann. of Math. (2) **42** (1941), 994–1024; MR **3**, 205] by replacing the norm by a sequence of pseudonorms. {Reviewer's comment: Theorem 6, which asserts that within the class of locally compact, σ-compact spaces, Ω is determined by the lattice structure of $C_0(\Omega)$, was proved by Shirota [Osaka Math. J. **4** (1952), 121–132; MR **14**, 669]. In fact, Shirota obtained the same conclusion for a very wide class of spaces, including all σ-compact spaces and essentially all metrizable spaces.} *M. Jerison* (Princeton, N.J.).

Arsove, Maynard G. Bases semblables et isomorphismes dans les espaces de Fréchet. C. R. Acad. Sci. Paris **246** (1958), 1143–1145.

Let V be a real (or complex) F-space. If $\{x_n\}$ and $\{y_n\}$ are two sequences in V, one writes $\{x_n\}\sim\{y_n\}$ if the series $\sum a_nx_n$ and $\sum a_ny_n$ are always either both convergent or both divergent. If $\{x_n\}$ is a basis for V, and T is an automorphism of V, then $y_n=Tx_n$ defines another basis $\{y_n\}$ with $\{y_n\}\sim\{x_n\}$. Theorem: If $\{y_n\}$ is any basis for V with $\{y_n\}\sim\{x_n\}$, then there is an automorphism T with $y_n=Tx_n$. Moreover, a similar result holds for absolute bases, enabling one to decide whether two absolute bases are connected by an automorphism. [In this connection, see also W. F. Newns, Philos. Trans. Roy. Soc. London. Ser. A. **245** (1953), 429–468; MR **14**, 968.] *R. C. Buck.*

Björck, Göran. The set of extreme points of a compact convex set. Ark. Mat. **3** (1958), 463–468.

For a subset X of a linear space, let $E(X)$ denote the

set of all extreme points of X and $H(X)$ the convex hull of X. Sets of the form $E(K)$ for some convex K are proved to be exactly those which are convexly independent — i.e., S has that form if and only if $S \cap H(A) \subset A$ for all $A \subset S$. This result is used as a lemma in proving that a subset S of Euclidean n-space has the form $E(K)$ for some compact convex K if and only if S is convexly independent, $\bar{S}$ is compact, and $\bar{S} \subset H(S)$. For such an S, it follows by Caratheodory's theorem that $\bar{S} \subset H_{n+1}(S)$, but the author shows that in fact $\bar{S} \subset H_{n-1}(S)$, thus extending the result that $E(K)$ is closed for two-dimensional compact convex K.

From the Krein-Milman theorem it follows that if K is a compact convex set in a locally convex space L, and $S = E(K)$, then each point of K is the barycenter of some measure on S. A recent result of Choquet [C. R. Acad. Sci. Paris **243** (1956), 699–702; MR **18**, 219] asserts that if L is metrizable then in fact $K = H_e(S)$, where $H_e(S)$ is the set of all points x of the form $x = \int_{\bar{S}} t\,d\mu(t)$, for μ a non-negative Radon measure on S with $\mu(\bar{S}) = 1$ and $\mu(\bar{S} - S) = 0$. In the present paper, Choquet's theorem is applied to show that a subset S of a locally convex complete metric space has the form $E(K)$ for some compact convex K if and only if $\bar{S}$ is compact, $\bar{S} \subset H_e(S)$, and $S \cap H_e(A) \subset A$ for all $A \subset S$. Some interesting refinements and examples are also given. *V. Klee* (Copenhagen).

Fan, Ky. Existence theorems and extreme solutions for inequalities concerning convex functions or linear transformations. Math. Z. **68** (1957), 205–216.

It is first proved, using a minimax theorem, that if K is a compact convex set in a topological vector space and $\{f_\nu\}_{\nu \in I}$ is a family of lower semi-continuous convex functions on K, then the system of inequalities $f_\nu(x) \leqq 0$ $(\nu \in I)$ is satisfied by some x in K if and only if $\operatorname{Min}\{\sum_{i=1}^{n} \lambda_i f_{\nu_i}(x) : x \in K\} \leqq 0$ for each finite subset $\nu_1, \cdots, \nu_n$ of I and for arbitrary non-negative λ_i. It is then proved that if the set S of vectors x in K satisfying the system of inequalities is non-empty, then for each upper semi-continuous concave function g defined on K,

$$\operatorname{Max}\{g(x) : x \in S\} = \operatorname{Inf} \operatorname{Max}\{g(x) - \sum_{i=1}^{n} \lambda_i f_{\nu_i}(x) : x \in K\},$$

where the Inf is taken over all finite sets $\nu_1, \cdots, \nu_n$ of indices and $\lambda_1, \cdots, \lambda_n$ of non-negative real numbers. A more precise result is also given for a finite system of inequalities. As a simple application of the first theorem, a proof is given of a theorem of Hardy, Littlewood and Pólya [Inequalities, Cambridge, 1934] on the existence of a doubly stochastic matrix mapping a finite sequence on to a majorized sequence. It is also used to prove some theorems on the existence of approximate solutions of linear operational equations. For example, let C be a convex subset of a Banach space X that contains the origin and is Euclidean closed in the sense that each continuous linear mapping of X into a Euclidean space maps C onto a closed set. It is proved that if A is a compact linear operator in X, $y_0 \in X$ and $\lambda \neq 0$, then there exists a vector x in X such that $Ax - \lambda x - y_0 \in C$ if and only if $1 + f(y_0) \geqq 0$ for every continuous linear functional f with $A^*f - \lambda f = 0$ and $f(c) \leqq 1$ $(c \in C)$. *F. F. Bonsall.*

Weston, J. D. The decomposition of a continuous linear functional into non-negative components. Math. Scand. **5** (1957), 54–56.

Let X be a real topological vector space in which a subset C is distinguished such that $C + C \subseteq C$ and $\alpha C \subseteq C$ $(\alpha \geqq 0)$. C determines a partial ordering of X in the usual sense that $u \leqq v$ means $v - u \in C$. The author proves that a continuous linear functional f on X has a decomposition $f = f_1 - f_2$, with f_1 and f_2 continuous linear functionals non-negative on C, if and only if there exists an open convex neighbourhood U of the origin such that $f(x) < 1$ whenever $0 \leqq x \leqq u$ and $u \in U$. *F. F. Bonsall.*

Bonsall, F. F. Linear operators in complete positive cones. Proc. London Math. Soc. (3) **8** (1958), 53–75.

This paper extends and unifies previous work of the author [J. London Math. Soc. **30** (1955), 133–144, 144–153; MR **16**, 936] and of Krein and Rutman [Uspehi Mat. Nauk (N.S.) **3** (1948), no. 1(23), 3–95; MR **10**, 256; **12**, 341]. Let X^+ be a non-trivial complete positive cone in a normed vector space over the real field; denote by (X^+, X^+) the linear operators on $X^+ - X^+$ which map X^+ into itself, and are continuous on X^+. The basic concept of the present paper is that of the partial spectral radius μ: If $T \in (X^+, X^+)$ and $p(T) = \sup\{\|Tx\| : x \in X^+, \|x\| \leqq 1\}$, then $\mu(T) = \lim_{n \to \infty} [p(T^n)]^{1/n}$. For $T \in (X^+, X^+)$, compactness in X^+ is defined in the obvious way.

Theorem 1. If T is compact in X^+, and $\mu > 0$, there is a non-zero $u \in X^+$ such that $Tu = \mu u$. Theorem 2: If X^+ is normal then $\mu x - Tx = y$ fails to have a solution $x \in X^+$ for some $y \in X^+$. The proofs are elementary in character, and involve nothing worse than the series expansion of resolvent operators, and the selection from given sequences of suitably chosen subsequences. From these two theorems, there follow various known results, of similar type, involving the spectral radius ρ instead of μ. The transition is effected by observing that under the hypotheses of these theorems, we have $\rho = \mu$. The paper contains a number of examples and counter-examples, some of which are of considerable independent interest.

In the last section a topology is described which has applications to operator theory. Let $\mathfrak{X}$ be an abelian semigroup with zero, complete with respect to an invariant metric δ; denoting by U_ε the subset $\{x : \delta(x, 0) < \varepsilon\}$ of $\mathfrak{X}$, the open sets of this topology of $\mathfrak{X}$ are those sets G such that for every $x \in G$ there exists $\varepsilon > 0$ with $x + U_\varepsilon \subset G$. In particular, let $\mathfrak{X}$ be a semi-space in a normed vector space (including the case of a positive cone). The fact that $\mathfrak{X}$ is of the second category is exploited to give theorems (3 and 4) on uniform boundedness, and inversion of linear mappings, generalising the familiar results for Banach spaces. These results are then applied as follows. A linear mapping T of $\mathfrak{X} - \mathfrak{X}$ into itself is partially bounded if TU_1 is bounded. The partial spectrum $\sigma(T)$ of an operator T with $T\mathfrak{X} \subset \mathfrak{X}$ is the set of $\lambda \geqq 0$ such that $\lambda I - T$ fails to be a 1-1 map of $\mathfrak{X}$ onto a set containing $\mathfrak{X}$. Theorem 5: Let $\lambda \geqq 0$, $\lambda \notin \sigma(T)$. Then $\lambda I - T$ is a partially bounded linear map of $\mathfrak{X} - \mathfrak{X}$ onto itself, and $(\lambda I - T)^{-1}$ is a partially bounded linear map of $\mathfrak{X}$ into itself. Theorem 6: $\sigma(T)$ is a subset of $[0, \mu(T)]$, closed on the right. If $\mathfrak{X}$ is a non-trivial normal positive cone then $\sigma(T) = [0, \mu(T)]$. *J. H. Williamson* (Belfast).

Edwards, R. E. Representation theorems for certain functional operators. Pacific J. Math. **7** (1957), 1333–1339.

Let $\mathfrak{F} = \{f(t) | t > 0,\ f(t)$ measurable, locally integrable$\}$. For f in $\mathfrak{F}$, $s \geqq 0$, let $U_s(f(t)) = f(t-s)\chi_{(0,\infty)}(t-s)$. Assume T maps $\mathfrak{F}$ into $\mathfrak{F}$ and commutes with U_s, all $s \geqq 0$. If T is continuous in the topology on $\mathfrak{F}$ defined by the seminorms $p_n(f) = \int_0^n |f(t)|\,dt$, then there is a Radon measure μ

on $[0, \infty)$ such that $T(f)=\int_0^t f(t-s)d\mu(s)$. Each such Radon measure induces a continuous T.

Let $\mathfrak{B}_n=\{f(t)|f \text{ in } \mathfrak{F},\ q_n(f)=\int_0^\infty e^{-nt}|f(t)|dt<\infty\}$. Each $\mathfrak{B}_n$ is a Banach space relative to the norm q_n. Let $\mathfrak{B}=\bigcup_{n=1}^\infty \mathfrak{B}_n=\lim_{n\to\infty}\mathfrak{B}_n$. If T maps $\mathfrak{B}$ into $\mathfrak{B}$, commutes with U_s, $s\geqq 0$, and if T is continuous relative to the inductive limit topology of $\mathfrak{B}$, then T is again representable as a convolution with a Radon measure μ which satisfies $\int_0^\infty e^{-nt}d|\mu|(t)<\infty$ for some n (depending on T).

B. R. Gelbaum (Minneapolis, Minn.).

Browder, Felix E. Non-linear functional equations in locally convex spaces. Duke Math. J. **24** (1957), 579–589.

The first section of the paper deals with several "invariance of domain" theorems. We quote one of them: Let X be a linear Hausdorff space, and U an open convex symmetric neighborhood of the origin in X. Let $p_U(x)$ be the seminorm corresponding to U, i.e., $p_U(x)=\inf\{\lambda|\lambda>0, x\in\lambda U\}$, and $N_U=\{x|x\in X, p_U(x)=0\}$. A continuous mapping C of $\bar{U}$ into X is called completely continuous if $C(\bar{U})$ is contained in a compact convex subset of X; C is called π-completely continuous with respect to U if in addition C is constant on the equivalence classes modulo N_U and if the mapping $U/N_U\to X/N_U$ obtained by the composition of C with the natural projection of X onto X/N_U is continuous in the p_U norm. Let $T(x)=x+C(x)$ and define the corresponding map, T_U, of $\bar{U}/N_U$ into X/N_U, by setting $T_U(\tilde{x})=\tilde{x}+\tilde{C(x)}$, where $\tilde{x}$ denotes the equivalence class mod N_U to which x belongs. The theorem to be quoted states: if T_U is one-to-one then $T(U)$ is open in X.

The second section deals with a family of mappings of a subset of a locally convex linear space X into a linear convex space Y, and the concept of a permissible family T_t $(0\leqq t\leqq 1)$ of such mappings is introduced for whose rather technical definition we refer to the paper. Using the results of the first section the following theorem is proved for such a family: if $T_0(x)=0$ has exactly m solutions $x_1^0, \cdots, x_m^0$ then there exist m continuous curves $x_i=x_i(t)$ $(i=1, \cdots, m; 0\leqq t\leqq 1)$ such that i) $x_i(0)=x_i^0$, ii) $x_i(t)\neq x_j(t)$ for $i\neq j$, iii) $T_t(x_i(t))=0$, iv) the curves $x_i(t)$ contain all solutions of $T_t(x)=0$.

The third section of the paper deals with theorems giving sufficient conditions for certain maps to be covering maps. We quote again one of them: let X and Y be locally convex linear spaces, the latter being infinite-dimensional, and T a continuous map of X into Y. A point x of X is called regular (with respect to T) if there exists a neighborhood N of x such that the restriction of T to N is a homeomorphism of N on a neighborhood of $T(x)$. Let R denote the set of regular points, and B its complement in X. Then for T to be a covering map it is sufficient that the following three assumptions are satisfied i) $T(R)-T(B)$ is not empty, ii) T is a closed mapping, iii) $T(B)$ is compact.

The content of the paper consists partly in generalizing results to be found in earlier literature, partly in making such results more precise, partly in giving proof for statements to be found in the earlier literature for which no proof seems to have been published. A detailed description of the relation of the present paper to earlier ones would, therefore, be rather lengthy and we refer in this respect to the author's own remarks, particularly to the introduction and to the "Remark" on top of the page 583. Let us just mention that the first two sections are based on results and ideas of the mapping degree theory which are connected with the names of Schauder, Leray and Nagumo while the third one is based on the general topological theory of covering spaces and an earlier paper of the author [same J. **21** (1954), 329–336; MR **15**, 978]. Though the paper deals with uniqueness and multiplicity questions no use is made of results known concerning "local branching", and the invertibility or even existence of a differential of the mappings considered is nowhere assumed.

E. H. Rothe (Ann Arbor, Mich.).

Banach Spaces, Banach Algebras, Hilbert Spaces

Kreĭn, M. G. On Bari bases of a Hilbert space. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 333–341. (Russian)

A sequence $\{\psi_j\}$, $\psi_j\in H$, H a separable Hilbert space, is called a "Bari base" by the author if ψ_j are "ω-linearly independent" (i.e. $\sum_1^\infty c_j\psi_j=0$ only if all $c_j=0$) and there exists a complete orthonormal system (c.o.s.) $\{\varphi_j\}$ such that $\sum_1^\infty\|\psi_j-\varphi_j\|^2<\infty$ [cf. N. Bari, Mat. Sb. N.S. **14(56)** (1944), 51–108; Moskov. Gos. Univ. Uč. Zap. **148**, Mat. **4** (1951), 69–107; MR **6**, 267; **14**, 289]. It is proved that a complete sequence $\{\psi_j\}$, $\|\psi_j\|=1$, is a Bari base if and only if the Gram determinant $\lim_{m\to\infty}\det\{(\psi_j, \psi_k)\}_1^m$ is positive (this result is closely related to Theorem 1 in the first of N. Bari's notes mentioned above); a complete sequence $\{\psi_j\}$ is a Bari base if and only if ψ_j are ω-linearly independent and $\sum_{j,k}|(\psi_j, \psi_k)-\delta_{jk}|^2<\infty$ [in connection with this condition cf. H. von Koch, Acta Math. **24** (1901), 89–122]. If $\{\psi_j\}$ is a Bari base, then, for any c.o.s. $\{\varphi_j\}$, $\sum_1^\infty\|\psi_j-\varphi_j\|^2\geqq\sum_1^\infty(\mu_n^{\frac{1}{2}}-1)^2$ where μ_n are eigenvalues of the Gram matrix $\{(\psi_j, \psi_k)\}_1^\infty$; the equality sign holds for a unique c.o.s. $\{\varphi_j\}$.

M. Katětov (Prague).

Semadeni, Z. Spaces of continuous functions. II. Studia Math. **16** (1957), 193–199.

Multiplicative linear functionals are constructed on certain spaces of equivalence classes of functions by exploiting the fact that these spaces are abstract (M)-spaces.

M. Jerison (Princeton, N.J.).

Stein, E. M.; and Weiss, G. Interpolation of operators with change of measures. Trans. Amer. Math. Soc. **87** (1958), 159–172.

Let M be an abstract set, $\mathfrak{M}$ a σ-algebra of subsets of M, and μ a non-negative measure on $\mathfrak{M}$. Let $\alpha_0(u)$ and $\alpha_1(u)$ be positive $\mathfrak{M}$-measurable functions, and let for E in $\mathfrak{M}$

$$\mu_t(E)=\int_E \alpha_0(u)^{(1-t)}\alpha_1(u)^t d\mu(u).$$

Let N, $\mathfrak{N}$, ν, $\beta_0(u)$, $\beta_1(u)$, and $\nu_t(F)$ be similarly defined. If T is a linear (or sublinear) transformation which is (i) bounded from $L^{p_0}(M, \mathfrak{M}, \mu_0)$ to $L^{q_0}(N, \mathfrak{N}, \nu_0)$ and (ii) bounded from $L^{p_1}(M, \mathfrak{M}, \mu_1)$ to $L^{q_1}(N, \mathfrak{N}, \nu_1)$ (here $1\leqq p_0, q_0, p_1, q_1\leqq\infty$), then (*) T is bounded from $L^{p_t}(M, \mathfrak{M}, \mu_t)$ to $L^{q_t}(N, \mathfrak{N}, \nu_t)$, where $p_t^{-1}=(1-t)p_0^{-1}+tp_1^{-1}$, $q_t^{-1}=(1-t)q_0^{-1}+tq_1^{-1}$. If α_0, α_1, β_0 and β_1 are identically 1, then the above result reduces to the classical Riesz-Thorin convexity theorem. If $p_0\neq p_1$ and $q_0\neq q_1$, then the general case can be obtained from the classical case by a simple mapping. Following Marcinkiewicz, A. Zygmund has given an extension of the classical Riesz-Thorin theorem in which the same conclucion (*) is drawn from assumptions weaker than (i) and (ii) [J. Math. Pures Appl. (9) **35** (1956), 223–248; MR **18**, 321]. The authors point out that the mapping mentioned above provides a similar

extension for the more general case described above. Applications are given.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Foiaş, C.; Gussi, Gh.; et Poenaru, V. L'étude de l'équation $du/d\tau=A(\tau)u$ pour certaines classes d'opérateurs non bornés de l'espace de Hilbert. Trans. Amer. Math. Soc. **86** (1957), 335–347.

The equation of the title is studied under various versions of the strong hypothesis that the Hilbert space H in which the operator family $A(\tau)$ acts has a dense subspace of the form $\bigcup_{n=1}^{\infty} H_n$, each H_n being a space invariant under all the operators $A(\tau)$, the restrictions of $A(\tau)$ to H_n being bounded and uniformly continuous in the parameter τ. Solutions of the indicated equation with prescribed initial vectors within any space H_n are then readily constructed. Supplementary arguments are given to establish conditions in which this set of solutions can be extended to a set of solutions with arbitrary initial vectors in H. The condition

$$\mathrm{Re}\,\langle A(\tau)x, x\rangle \leq \mu(\tau)\|x\|^2,$$

where $\int \mu(\tau)d\tau<\infty$, is shown to be sufficient in order that solutions with arbitrary initial vectors in H exist.

J. Schwartz (Upton, N.Y.).

Plans, Antonio. A system of axioms for the ring of real bounded infinite matrices. Collect. Math. **9** (1957), 35–40. (Spanish)

The set of real infinite matrices which represent bounded operators on l_2 is characterised as the maximal system S satisfying certain conditions. The vital axioms are that if $A, B \in S$ then $\sum_j a_{ij}b_{jk}$ converges absolutely for all i, k, and $AB \in S$; and that if $A'A \in S$ then $A \in S$.

J. H. Williamson (Cambridge, England).

Plans, Antonio. A lattice structure of the ring of real bounded infinite matrices. Collect. Math. **9** (1957), 87–104. (Spanish)

The author defines equivalence-classes, and certain operations of union and intersection, in the ring of infinite matrices which are bounded as operators on l_2. In terms of these he defines ideals, which turn out to be the familiar ring-ideals. Various elementary properties are discussed, and a mapping ∂ of the right (left) ideals into the left (right) ideals is introduced. An ideal $\mathfrak{J}$ is called closed if $\partial\partial\mathfrak{J}=\mathfrak{J}$; for closed ideals a fuller theory is developed, including a duality principle. Finally, bounded infinite matrices are classified according to the principal ideals which they generate. Misprint: p. 96, l. 14, left-hand relation should be $XY=0$. *J. H. Williamson.*

Thoma, Elmar. Zur Reduktionstheorie in allgemeinen Hilbert-Räumen. Math. Z. **68** (1957), 153–188.

The main part of this paper is concerned with the reduction theory for a commutative self-adjoint ring $\mathfrak{R}$ of bounded operators on a Hilbert space $\mathfrak{H}$ (not necessarily separable) by means of a direct integral of Hilbert spaces. This means there exists a measure space $(\mathcal{M}, \mathcal{B}, \mu)$ and Hilbert spaces $\mathfrak{H}(x)$ $(x \in \mathcal{M})$ such that $\mathfrak{H}=\int \mathfrak{H}(x)d\mu(x)$ and every $A \in \mathfrak{R}$ may be written as $(Af)(x)=a(x)f(x)$, where $a(x)$ is a bounded measurable complex-valued function. Further, for every $A \in \mathfrak{R}'$, the commutor of $\mathfrak{R}$, we have $(Af)(x)=A(x)f(x)$ where $A(x)$ is a bounded operator in $\mathfrak{H}(x)$ a.e. These results for general Hilbert spaces have already been obtained by R. Godement [Ann. of Math. (2) **53** (1951), 68–124; MR **12**, 421] and I. E. Segal [Mem. Amer. Math. Soc. no. **9** (1951); MR **13**, 472]; for separable Hilbert spaces the results are due to J. von Neumann [Ann. of Math. **50** (1949), 401–485; MR **10**, 548]. The present author does not consider any reducibility questions.

The difference between this paper and those of Godement and Segal is one of method. Godement and Segal reduce the problem to the consideration of real-valued measurable functions through the use of positive linear functionals (states) on $\mathfrak{R}$. The present author proceeds directly. He considers a complete Boolean lattice $\mathfrak{M}$ of commuting projections containing the identity. By a well-known theorem of M. H. Stone, $\mathfrak{M}$ is isomorphic to a field $\mathfrak{K}$ of sets obtained in the following way: Let $\mathcal{M}$ be the set of all ultra-filters of $\mathfrak{M}$ and for any $E \in \mathfrak{M}$, M_E the class of all ultra-filters containing E, $\mathfrak{K}=\{M_E; E \in \mathfrak{M}\}$. Let $\mathfrak{K}$ be a basis of open sets for a topology on $\mathcal{M}$; then $\mathcal{M}$ becomes a totally disconnected space. As the field $\mathcal{B}$ on which μ is constructed take the Borel field formed by the open sets plus sets of the first category of $\mathcal{M}$. Every measurable function on $\mathcal{M}$ is then equal a.e. to a continuous function. For every $M \in \mathcal{B}$ there exists a unique $M_E \in \mathfrak{K}$ which differs from M by a set of the first category. Define $E(M)=E$ and this gives a spectral measure on $\mathcal{B}$. The measure μ is constructed from this spectral measure by means of the inner product on $\mathfrak{H}$. In this construction essential use is made of the Nakano-Plessner-Rohlin-Wecken multiplicity theory [cf. P. Halmos, Introduction to Hilbert Space, Chelsea, New York, 1951; MR **13**, 563]. Once the reduction theorem has been obtained a number of results concerning the ring generated by $\mathfrak{M}$ and its commutors can be obtained.

Using the direct integral decomposition of $\mathfrak{H}$ relative to $\mathfrak{R}$ the author shows how to extend $\mathfrak{R}$ to a commutative ring containing unbounded operators [cf. J. M. G. Fell and J. L. Kelly, Proc. Nat. Acad. Sci. U.S.A. **38** (1952), 592–598; MR **14**, 480; H. Nakano, Modern spectral theory, Maruzen, Tokyo, 1950; MR **12**, 419]. These are all the normal operators defined by $(Af)(x))=a(x)f(x)$ where $a(x)$ is a complex-valued measurable function on $\mathfrak{M}$ which is infinite only on a nowhere-dense set. The domain of A is the class of all f such that $\int|a(x)f(x)|^2d\mu(x)<\infty$.

The last two sections of this paper are concerned with some topological questions in $\mathcal{M}$. *A. Devinatz.*

Leptin, Horst. Zur Reduktionstheorie Hilbertscher Räume. Math. Z. **69** (1958), 40–58.

The author obtains known results in the reduction theory for a commutative self-adjoint ring $\mathfrak{R}$ of bounded operators on a Hilbert space $\mathfrak{H}$ [see the preceding review]. Although the author specifically notes that he lays no claim to originality of results, the paper is a neat introduction to the subject. No use is made of special theories of normed rings or operator algebras as in the papers of Godement and Segal, and the necessary multiplicity theory is developed in the paper.

The integration theory used throughout the paper is in the sense of Daniell. Let $\mathcal{P}$ be a complete Boolean lattice of commuting projections, not identically the null projection, and E its maximal element. $\mathcal{P}$ is said to be cyclic if there exists an $e \in \mathfrak{H}$ such that $Ee=e$ and $|Pe|<|e|$ for every $P \in \mathcal{P}$ which is different from E. $\mathcal{P}$ is called atomic if it contains exactly the two elements $\{0, E\}$, discrete if it is a direct sum of atomic lattices and dense if between any two comparable elements is a third. Since every $\mathcal{P}$ is a direct sum of cyclic ones, every $\mathcal{P}$ is a sum of a discrete and dense part, and the reduction theory for discrete lattices is trivial, it is enough to consider $\mathcal{P}$ as cyclic and dense. For such lattices it is shown that there exists a measure space (S, I) (I is the Daniell integral)

such that

$$E\mathfrak{H}=\sum_{\mu\leq\zeta}\mu L^2(S_\mu) \text{ and } \mathscr{P}=\sum_{\mu\leq\zeta}\mathscr{P}_\mu, \zeta=\dim E\mathfrak{H},$$

where $S_\mu\subseteq S$ and $\mathscr{P}_\mu$ is the complete Boolean lattice of projections corresponding to characteristic functions of measurable subsets of S_μ. The same result may then be obtained for a general $\mathscr{P}$ by decomposing $\mathscr{P}$ into its discrete and dense cyclic parts.

A. Devinatz (St. Louis, Mo.).

Leptin, Horst. Reduktion linearer Funktionale auf Operatorringen. Abh. Math. Sem. Univ. Hamburg **22** (1958), 98–113.

The author continues his study of the reduction theory for a commutative self-adjoint ring R of bounded operators on a (not necessarily separable) Hilbert space by methods initiated in the paper reviewed above. Essentially known results are obtained concerning the reduction of operators in the commutor ring R'.

As the title of the paper indicates, the author reduces the problem to the study of positive linear functionals on R, which is reminiscent of the work of Segal [Mem. Amer. Math. Soc. no. **9** (1951); MR **13**, 472] and Godement [Ann. of Math. (2) **53** (1951), 68–124; MR **12**, 421]. Another development very closely related to that of this paper may be found in two papers of M. Tomita [Mem. Fac. Sci. Kyūsyū Univ. A. **7** (1953), 129–168; Math. J. Okayama Univ. **3** (1954), 147–173; MR **15**, 632, 968].

A. Devinatz (St. Louis, Mo.).

Aurora, Silvio. Multiplicative norms for metric rings. Pacific J. Math. **7** (1957), 1279–1304.

This paper presents a systematic study of the possible types of metric rings R (associative and with unit) equipped with a norm satisfying the usual conditions for normed algebras (apart from that involving multiplication by scalars), when the inequality $\|ab\|\leq\|a\|\cdot\|b\|$ is strengthened in various ways. It therefore follows up and generalizes the work of Mazur, Gelfand, Lorch and many others relating to normed algebras. The results are too numerous to record in detail. Broadly speaking, they fall into two main groups. The first comprises various sets of conditions under which a metric ring R will be a division algebra and its norm multiplicative (an absolute value). For example (part of Theorem 10), one set of sufficient conditions is: (i) R be a Q-ring (i.e., the inversible elements form an open set G); (ii) $R\cap\mathscr{C}(0)$ be connected; (M2) $\|xy\|=\|x\|\cdot\|y\|$ for $x, y\in G$. Each of the second group of results gives a set of conditions under which R is forced to be isomorphic with the real field, the complex field, or the quaternions. As a sample (part of Theorem 14): sufficient conditions are that R be complete, that (M2) shall hold, and that the set H, formed of those $a\in R$ for which

$$\operatorname{Inf}\|ax\|/\|x\|>0, \quad \operatorname{Inf}\|xa\|/\|x\|>0$$

(infimum with respect to all $x\neq 0$ in R), be both connected and dense.

The results obtained for metric rings are specialised for (real or complex) Banach algebras in such a way as to yield theorems of Mazur [C. R. Acad. Sci. Paris **207** (1938), 1025–1027] and the reviewer [Proc. Cambridge Philos. Soc. **47** (1951), 473–474; MR **13**, 256].

R. E. Edwards (London).

See also: Sequences, Series, Summability: Davis. **Lie Groups and Algebras:** Naĭmark. **Topological Vector Spaces:** Bonsall; Edwards.

TOPOLOGY

General Topology

Borsuk, Karol. Über einige Probleme der anschaulichen Topologie. Jber. Deutsch. Math. Verein. **60** (1958), Abt. 1, 101–114.

This is an address, giving short historical summaries, motivations, and the present state, of the principal (non-algebraic) topological problems in the domains of interest of the author. The problems are grouped under (1) retract theory (centering around AR and ANR), (2) Cartesian products (representation of given spaces as Cartesian products), and (3) development of the topology of E^n by maps into spheres.

J. Dugundji.

de Vito, Luciano. Sulla connessione dei campi piani. Rend. Mat. e Appl. (5) **16** (1957), 297–314.

Es werden wohlbekannte Zusammenhangsfragen bei ebenen Gebieten mit sogenannten "elementaren Methoden" behandelt, da nach Ansicht des Verfassers andere Darstellungen nicht elementar genug sind oder nicht von genau den gleichen Definitionen ausgehen.

A. Pfluger (Zürich).

Molski, R. On symmetric products. Fund. Math. **44** (1957), 165–170.

It is shown that the symmetric square of a 2-cell is a 4-cell but the symmetric square of an n-cell for $n\geq 3$ is not a subspace of euclidean $2n$-space. Moreover, symmetric mth powers of a 2-cell for $m\geq 3$ are not subspaces of $2m$-space. The result on n-cells depends on homological conditions for embedding and on the presence of $S^n\times P^{n-1}$ in the symmetric square. As to the higher powers of a 2-cell, it is shown that they are not regularly connected, i.e. they contain open $2m$-cells which are not open sets.

J. Isbell (Seattle, Wash.).

Henriksen, Melvin. Some remarks on a paper of Aronszajn and Panitchpakdi. Pacific J. Math. **7** (1957), 1619–1621.

A topological property introduced by Aronszajn and Panitchpakdi [Pacific J. Math. **6** (1956), 405–439; MR **18**, 917] is shown to be equivalent in normal spaces to the property of being an F-space [Gillman and Henriksen, Trans. Amer. Math. Soc. **82** (1956), 366–391; MR **18**, 9]. Negative solutions of two problems proposed in the former paper follow almost at once.

J. Isbell.

Kerstan, Johannes. Eine charakteristische Eigenschaft der Lindelöfschen und der parakompakten Räume. Math. Nachr. **16** (1957), 391–392.

Some results on special bases in Lindelöf and paracompact spaces, using the concept of a totally open set (= inverse image of an open set under a continuous, real-valued function). For instance, in a regular Lindelöf space the collection of totally open sets is the only base which is closed under countable unions. A partial converse of this result is also given, as well as analogous results about paracompact spaces. *E. Michael.*

Choquet, Gustave. Une classe régulière d'espaces de Baire. C. R. Acad. Sci. Paris **246** (1958), 218–220.

Two well-behaved classes of spaces of the second category are introduced, as follows. A topological space is said to be siftable ("tamisable") if there is a partial ordering of the non-empty open sets which is (1) weaker than inclusion, (2) hereditary, in which (3) every non-empty open set has a predecessor, and (4) every descending sequence has a common point. If (3) can be strengthened to the statement that the predecessors form a basis, the space is called strongly siftable. The following results are announced. A metrizable space is siftable if and only if it is residual in its completion, or in any compactification; this is false ("only if" is false) for non-metrizable spaces. A metrizable space is strongly siftable if and only if it is an absolute G_δ; this is an open question for non-metrizable spaces. Both properties, siftable and strongly siftable, are local and are preserved by arbitrary Cartesian products. The former is inherited by residual subsets; it is an open question if the latter is inherited by G_δ subsets. *J. Isbell* (Seattle, Wash.).

Balcerzyk, S.; and Mycielski, Jan. On the method of category in analytic manifolds. Fund. Math. **44** (1957), 295–299.

The author verifies in two cases the conjecture of W. Sierpiński and F. Hausdorff that in a complete metric space E the sum of fewer than $2^{\aleph_0}$ nowhere dense subsets S is a border set in the space. The cases are: E=euclidean E^n and S=convex surface; E=real analytic space, S=real analytic surface. *S. Bochner.*

Kac, G. I. On completely regular spaces without complete uniform structures. Uspehi Mat. Nauk (N.S.) **12** (1957), no. 3(75), 329–332. (Russian)

It is proved that if D is a two-point space, $\tau>\aleph_0$, $x\in D^\tau$, $S=D^\tau-(x)$, then $D^\tau=\beta S$. From this lemma, the following theorem is obtained: if Q is a product of uncountably many completely regular (non-trivial) spaces, $x\in Q$, then $Q-(x)$ is not topologically complete (i.e. the topology of $Q-(x)$ cannot be generated by a uniformity under which $Q-(x)$ is a complete uniform space). *M. Katětov* (Prague).

Dyer, E.; and Hamstrom, M.-E. Completely regular mappings. Fund. Math. **45** (1958), 103–118.

If X and Y are metric spaces, a mapping f of X onto Y is completely regular if for each $y\in Y$ and $\varepsilon>0$, a $\delta>0$ exists such that for any $x\in Y$ with $\rho(x,y)<\delta$, there is a homeomorphism of $f^{-1}(y)$ onto $f^{-1}(x)$ which moves no point as much as ε. The authors establish general theorems about such mappings which show that they are closely related to, though usually less restrictive than, the projection mappings. For example, if X is complete, Y is finite-dimensional, and there is a set K, which is either discrete or a simple closed curve, such that each $f^{-1}(y)$, $y\in Y$, is homeomorphic with K, then (X, f, Y) is a locally trivial fibre space. Also, it turns out that if f is a 0-regular mapping, and there is a space K, which is either an i-cell or an i-sphere ($i\leqq 2$), such that each $f^{-1}(y)$, $y\in K$, is homeomorphic to K, then f is completely regular. From this it is shown that if q is a closed mapping of E^3 onto a metric space Y such that $q^{-1}(y)$, $y\in Y$, is a (compact) continuum lying in a horizontal plane and not separating that plane, then Y is homeomorphic to E^3. Another application is made to show that the only compact metric space which can be mapped 0-regularly onto an arc such that each point inverse is a 2-cell is the 3-cell. *G. T. Whyburn* (Charlottesville, Va.).

Bing, R. H. Upper semicontinuous decompositions of E^3. Ann. of Math. (2) **65** (1957), 363–374.

Let G be an upper semicontinuous decomposition of Euclidean 3-space E^3 into continua, with the hyperspace topology defined in the natural way. The author gives various conditions which are sufficient to ensure that G is homeomorphic to E^3. These are as follows. A: The complement of each element of G is homeomorphic to the complement of a point, G has only a countable number of non-degenerate elements, and the union of the elements of G is a G_δ-set. (The author has announced that if its third part is omitted, Condition A is no longer sufficient.) B: G has only a countable number of non-degenerate elements g, and each of these is star-like, in the sense that there is a point p of g such that each line through p intersects g in an interval. C: H has only a countable number of non-degenerate elements, and each of these is a tame arc. D: Each non-degenerate element of G is a vertical interval. (The author notes a later announcement of M.-E. Hamstrom and E. Dyer [Bull. Amer. Math. Soc. **62** (1956), 398–399; cf. article reviewed above] that the following condition is also sufficient. D′: Every element g of G lies in a horizontal plane E and each set E^3-g is homeomorphic to the complement of a point.) The author has given an example to show that if the complement of each element of G is homeomorphic to the complement of a point, it does not follow that G is homeomorphic to E^3 [Summary of Lectures and Seminars, Summer Institute on Set Theoretic Topology, Madison, 1955, pp. 40–47; also the paper reviewed below].

Finally, the following theorem is proved. Let M be a continuum which is the union of two solid horned spheres (in the sense of Alexander) intersecting in their boundaries. Then there is an upper semicontinuous decomposition G of E^3, such that each non-degenerate element of G is a tame arc which intersects every horizontal plane in at most one point, and such that G, compactified at infinity, is homeomoprhic to M. *E. E. Moise.*

Bing, R. H. A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3. Ann. of Math. (2) **65** (1957), 484–500.

The author gives an example of an upper-semicontinuous decomposition G, of three-space E^3, into points and tame arcs, such that G is not homeomorphic to E^3. This result solves (in the negative) a problem of G. T. Whyburn [Bull. Amer. Math. Soc. **42** (1936), 49–73]. The author shows further that the space G is not a (Euclidean) three-manifold.

It follows from a theorem of R. L. Wilder [Pacific J. Math. **7** (1957), 1519–1528; MR **19**, 1188] that, since the natural pro-

jection $\pi: E^3 \to G$ is monotonic in every dimension, G must be a generalized (homological) three-manifold. And it has been shown by M. L. Curtis and R. L. Wilder [Bull. Amer. Math. Soc. **62** (1956), 180] that such a G must also be a homotopy manifold. Thus the stated property of G is an extremely delicate property. The construction of the example, and the preliminary steps in the proof, are too technical for a brief summary. *E. E. Moise.*

Fort, M. K., Jr. A note concerning a decomposition space defined by Bing. Ann. of Math. (2) **65** (1957), 501–504.

In the paper reviewed above, R. H. Bing has given an example of an upper-semicontinuous decomposition G of 3-space into points and tame arcs such that the associated hyperspace is not homeomorphic to 3-space. Let H be the set of all nondegenerate elements of G. The author shows that G can be constructed in such a way that each element of H is a broken line with exactly two breaks (i.e., is the union of three linear intervals). Given two parallel planes, we can require that each element of H has exactly one "break" in each of these planes. Finally, given $\theta > 0$, we can require that no element of H contain a broken line segment with an angle differing from a straight angle by more than θ. *E. E. Moise* (Ann Arbor, Mich.).

Curtis, M. L. An imbedding theorem. Duke Math. J. **24** (1957), 349–351.

Although it was known from joint work of the author and the reviewer that a 3-dimensional generalized closed manifold (i.e., 3-gcm) which (1) is imbeddable in E^4, or (2) satisfies certain local homotopy conditions suggested by Griffiths [Michigan Math. J. **2** (1954), 61–89; MR **16**, 159], is not necessarily a topological manifold in the classical sense, it was not known whether a 3-gcm which satisfies both (1) and (2) must be a topological manifold. The present paper shows that the answer is still negative, in that a quotient space of the 3-sphere given by Bing [paper reviewed second above], which satisfies condition (2) and is not locally euclidean, is imbeddable in E^4. *R. L. Wilder* (Ann Arbor, Mich.).

See also: Foundations, Theory of Sets, Logic: Bruns. **Fields, Rings:** Gillman. **Measure, Integration:** Cesari; Mickle and Neugebauer; Neugebauer. **Programming, Resource Allocation, Games:** Reichbach.

Algebraic Topology

★ **Aleksandrov, Pavel Sergeevič. Topologia combinatoria.** Manuali: serie di matematica. Edizioni Scientifiche Einaudi, Turin, 1957. xviii+750 pp. 10000 Lire.

A translation from the Russian original reviewed in MR **10**, 55.

Kodama, Yukihiro. On a right inverse mapping of a simplicial mapping. Proc. Japan Acad. **33** (1957), 136–138.

Let X, Y be simplicial complexes and $f: X \to Y$ a simplicial map. Homological criteria are given for the existence of a simplicial map $g: Y \to X$ such that $fg =$ identity. One of these is as follows: Let $\tilde{Y}$ be the barycentric subdivision of Y and $\tilde{X}$ the barycentric subdivision of $\tilde{X}$ rel $f^{-1}(Y^0)$ (i.e., simplexes of $f^{-1}(Y^0)$ are not subdivided; Y^i denotes the i-section of Y). g exists if the homomorphism of the homology-sequence of the pair $(\tilde{X}^1, X^0)$ into that of $(\tilde{Y}^1, \tilde{Y}^0)$ induced by f has a right inverse. *V. Gugenheim* (Baltimore, Md.).

Wilder, R. L. Monotone mappings of manifolds. Pacific J. Math. **7** (1957), 1519–1528.

Let $f: A \to B$ be a map of Hausdorff spaces with A compact, and let F be a fixed field. Call f "q-monotone" whenever, for all $b \in B$ and $r \leq q$, $H_r(f^{-1}(b)) = 0$ using Čech homology with coefficients in F. The author considers the case when A is also an n-dimensional generalised manifold over F (n-gcm) and B is non-degenerate (i.e., B has at least two points). Th. 1: "Let A be an orientable n-gcm and $f: A \to B$ an $(n-1)$-monotone mapping of A onto an at most n-dimensional non-degenerate Hausdorff space B. Then B is an orientable n-gcm of the same homology type as A." The homology type follows from the Vietoris mapping theorem (a global result) and the main point of the author's theorem is the fact that f preserves the local homology of A. The theorem is one way in which to generalise R. L. Moore's theorem that if $f: A \to B$ is a 0-monotone map of the 2-sphere onto a non-degenerate B, then B is homeomorphic to A (by some other map perhaps). A second theorem proved by the author is that if, in Th. 1, B is assumed only finite-dimensional, but f is $(n-1)$-monotone using integer coefficients, then the conclusion of Th. 1 still holds with coefficients still in the field F. The two theorems are proved using several interesting technical lemmas. (Forms of these theorems hold for locally compact, locally orientable spaces assuming orientability of the sets $f^{-1}(b)$, $b \in B$, as announced by the author in Bull. Amer. Math. Soc. **63** (1957), 292.)

H. B. Griffiths (Bristol).

Moore, John C. The double suspension and p-primary components of the homotopy groups of spheres. Bol. Soc. Mat. Mexicana (2) **1** (1956), 28–37.

Let S^{n+1} be the $(n+1)$-dimensional sphere. H. Cartan (unpublished) has computed the stable p-primary components of the homotopy groups $\pi_{2m+k}(S^{2m+1})$ over the range of dimensions for $k < 2(p+1)(p-1)-2$, $m > p$, where p is a prime. In this paper the author computes the non-stable homotopy groups $\pi_{2m+k}(S^{2m+1})$ for $2(p+1)(p-1)-2 \geq k > 1$ and for $p \geq m \geq 1$. {H. Toda has also obtained a little more extensive results including those of Cartan and the author; see the following review.} For that, he studies the double suspension E^2: $\pi_q(S^{n-1}) \to \pi_{q+2}(S^{n+1})$ for the case when n is even. Let $\Omega(S^{n+1})$ be the space of loops in S^{n+1} and let $\Omega^2(S^{n+1})$ be the space of loops in $\Omega(S^{n+1})$. Then the Pontrjagin ring $H_*(\Omega^2(S^{n+1}), Z_p)$ is calculated first, where Z_p is the group of integers mod p, and p is an odd prime. By using this he extends the known results on the homotopy groups of the pair $(\Omega^2(S^{n+1}), S^{n-1})$, and studies the inclusion homomorphism $\iota: \pi_q(S^{n-1}) \to \pi_q(\Omega^2(S^{n+1}))$, which gives the necessary information about the double suspension. *H. Uehara.*

Toda, Hirosi. On the double suspension E^2. J. Inst. Polytech. Osaka City Univ. Ser. A. **7** (1956), 103–145.

This paper establishes a close relation between p-primary components of the homotopy groups of the spheres S^{n-1}, S^{n+1}, S^{pn-1} and S^{pn+1}, where p is an odd prime and n is even. Similar results were obtained by the reviewer [Ann. of Math. (2) **65** (1957), 74–107; MR **18**, 662] for the case $p=2$, which is altogether simpler. Toda regards his work as a study of the double (i.e., iterated) suspension $E^2: \pi_{i-1}(S^{n-1}) \to \pi_{i+1}(S^{n+1})$. Among various

applications of the main theorem he proves that the cokernel of E^2 contains no elements of order p^3, and hence deduces that the homotopy groups of S^{n+1} contain no elements of order p^{n+1}.

We begin by embedding S^{n-1} in $\Omega^2(S^{n+1})$, the space of singular 2-spheres, so that we have an exact sequence

$$\cdots\to\pi_{i-1}(S^{n-1})\xrightarrow{E^2}\pi_{i+1}(S^{n+1})\to\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1})\to \pi_{i-2}(S^{n-1})\to\cdots.$$

Then certain maps are constructed, by procedures too complicated to explain in review, and the relative homotopy groups are analysed by the technique of spectral sequences. This produces the main theorem: that there exists a sequence

$$\cdots\to\pi_{i+2}(S^{pn+1})\xrightarrow{\Delta}\pi_i(S^{pn-1})\to\pi_{i-1}(\Omega^2(S^{n+1}), S^{n-1})\to \pi_{i+1}(S^{pn+1})\to\cdots$$

which is exact on the p-primary components (n even, p odd, $i>pn-1$). The homomorphisms in the sequence are not easy to describe. They do, however, transform functorially. Moreover, Toda shows that the composition

$$\pi_i(S^{pn-1})\xrightarrow{E^2}\pi_{i+2}(S^{pn+1})\xrightarrow{\Delta}\pi_i(S^{pn-1})$$

coincides with the endomorphism induced by a map of degree p.

The proof of the theorem involves the reduced product complex $S_\infty{}^n=S^n\cup e^{2n}\cup e^{3n}\cup\cdots$ which can be regarded as a homotopy equivalent of $\Omega(S^{n+1})$. Consider the subcomplex $S_{p-1}{}^n=S^n\cup e^{2n}\cup\cdots\cup e^{(p-1)n}$, where n is even and p prime. Toda shows that $\pi_i(S_\infty{}^n, S_{p-1}{}^n)$ and $\pi_{i+1}(S^{pn+1})$ have isomorphic p-primary components, likewise $\pi_i(\Omega(S_{p-1}{}^n), S^{n-1})$ and $\pi_{i+1}(S^{pn-1})$. By combining these two relations his main result is obtained.

The proofs are full of interesting details, such as a discussion of the way that the pn-cell of $S_\infty{}^n$ is attached to $S_{p-1}{}^n$. There are also some more general theorems, including a useful set of conditions (in terms of cohomology) which ensures that a map $(X, A)\to(Y, y_0)$ should induce isomorphisms of homotopy groups as if it were a fibre map.

Some of these results were announced by Toda in C. R. Acad. Sci. Paris **240** (1955), 42–44 [MR **16**, 846]. For a limited range of dimensions they overlap, to some extent, with the contents of a paper by Moore [reviewed above]. This applies particularly to the appendix, which gives examples to show how a complete stem of homotopy groups can be deduced from knowledge of the stable groups, in favourable cases. Shortly after publication Toda informed the reviewer that some of these illustrations are upset by the presence of an error in the supposed values of the stable groups concerned. But this is a minor blemish in work which has done much to strengthen our hold on the structure of the homotopy groups of spheres.

I. M. James (Oxford).

Granas, A. On the theory of Borsuk's cohomotopy groups. Fund. Math. **44** (1957), 159–164. (Russian)

In this note the author proves some theorems on cohomotopy groups which lead to a demonstration of the Phragmen-Brouwer theorem: If A, B are closed subsets of the $(n+1)$-sphere S_{n+1} with $\dim A\cap B\leq n-2$ and if $b_0(S_{n+1}\setminus F)$ is the number of components into which F separates S_{n+1}, then $b_0(S_{n+1}\setminus(A\cup B))=b_0(S_{n+1}\setminus A)+b_0(S_{n+1}\setminus B)-1$. Let X be compact, $\dim X<2n-1$, and A closed in X. The author writes $B_n(X)$ for the nth cohomotopy group of X, $H_n(X, A)$ for the subgroup of $B_n(X)$ consisting of classes of maps $f:X\to S_n$ such that $f|A$ is null-homotopic, and $G_n(A, X)$ for the subgroup of $B_n(A)$ consisting of classes of maps $f:A\to S_n$ admitting extension to X. If b_n, h_n, g_n are the ranks of $B_n(X)$, $H_n(X,A)$ and $G_n(A, X)$, then Theorem 1 asserts that $b_n=h_n+g_n$. {The reviewer remarks that this theorem follows immediately from the fact that $H_n(X, A)$ is the kernel and $G_n(A, X)$ the image of the restriction homomorphism $\rho_A:B_n(X)\to B_n(A)$.} If A, B are closed subsets of X, the author writes $P_n(A, B)$ for $H_n(A\cup B, A)\cap H_n(A\cup B, B)$ and p_n for the rank of P_n, and proves (Theorem 2) that $h_n(A\cup B, A\cap B)=h_n(A, A\cap B)+h_n(B, A\cap B)+p_n$. {The reviewer offers this proof: $B_n(A\cup B/A\cap B)=B_n(A/A\cap B)\oplus B_n(B/A\cap B)$. Thus, in the above notation with $X=A\cup B$, $\rho_{A\cap B}{}^{-1}(0)=\rho_A{}^{-1}(0)+\rho_B{}^{-1}(0)$, whence $h_n(A\cup B, A\cap B)=h_n(A\cup B, A)+h_n(A\cup B, B)-p_n$. By considering the cohomotopy homomorphisms induced by the inclusion $(B,A\cap B)\subset(A\cup B,A)$ it readily follows that $h_n(A\cup B, A)=h_n(B, A\cap B)+p_n$, whence the theorem.} The Phragmen-Brouwer theorem now follows as a very special case, using the 'duality' theorem $b_0(S_{n+1}\setminus F)=b_n(F)+1$.

P. J. Hilton (Ithaca, N.Y.).

Papakyriakopoulos, C. D. On the ends of the fundamental groups of 3-manifolds with boundary. Comment. Math. Helv. **32** (1957), 85–92.

The number e_M of ends of the fundamental group of a compact 3-dimensional manifold M is always either 0, 1, 2 or ∞ [Hopf, Comment. Math. Helv. **16** (1944), 81–100; MR **5**, 272]. The problem of finding the exact value of e_M was solved by Specker [ibid. **23** (1949), 303–332; MR **11**, 451] for the closed manifolds M. Completing some previous work [Ann. of Math. (2) **62** (1955), 293–299; MR **19**, 976] the author now solves the problem for manifolds whose boundaries are orientable surfaces. It is found that $e=1$ if M is aspherical and such that for any component F of the boundary the injection $\Pi_1(F)\to\Pi_1(M)$ is an isomorphism, $e=2$ if M is aspherical with connected boundary F of genus 1 and such that the injection $\Pi_1(F)\to\Pi_1(M)$ is not an isomorphism, $e=\infty$ otherwise. In the final paragraph the author shows how some of the argument can be shortened by making use of Dehn's lemma [ibid. **66** (1957), 1–26; MR **19**, 761] and the loop theorem [Proc. London Math. Soc. (3) **7** (1957) 281–299; MR **19**, 441].

R. H. Fox.

Noguchi, H. Errata: On regular neighbourhoods of 2-manifolds in 4-Euclidean space. I. Osaka Math. J. **9** (1957), 241–242.

The article is in same J. **8** (1956), 225–242 [MR **18**, 920].

Kondo, Koiti. On knotgroups of parallelknots. Kōdai Math. Sem. Rep. **9** (1957), 67–68.

The group of a knot k that can be drawn on the boundary of a closed tubular neighborhood V of some other knot was derived in Reidemeister's book [Knotentheorie, Springer, Berlin, 1932] by an unduly cumbersome process. The short snappy proof, using the "addition theorem of fundamental groups" applied to the decomposition of $R-k$ into $(R-V)+(V-k)$ and $V-k$, is exhibited here by the author.

R. H. Fox (Princeton, N.J.).

Weier, Joseph. Struttura dimensionale delle soluzioni di una classe di trasformazioni. Ann. Univ. Ferrara. Sez. VII. (N.S.) **5** (1955–1956), 49–58 (1957).

Let Φ be a homotopy class of mappings $Q\to S$, where S is an n-sphere and Q a compact simplicial manifold of dimension $n+1$. Let p be a point of Q. The author gives

conditions which ensure the existence of a mapping $g \in \Phi$ such that $g^{-1}(p)$ is zero-dimensional.

P. A. Smith.

See also: Topological Groups: Conner and Floyd. **Lie Groups and Algebras:** Adler. **General Topology:** Molski; Bing; Curtis. **Convex Domains, Integral Geometry:** Stein.

GEOMETRY

Geometries, Euclidean and Other

Schütte, Kurt. Der projektiv erweiterte Gruppenraum der ebenen Bewegungen. Math. Ann. **134** (1957), 62–92.

Das Ziel des Verf. ist, im Anschluß an eine frühere Arbeit [Math. Ann. **132** (1956), 43–62; MR **18**, 664] und an Arbeiten u.a. von A. Schmidt [ibid. **118** (1943), 609–625; J. Reine Angew. Math. **186** (1949), 230–240; MR **6**, 13; **12**, 524], F. Bachmann [Math.-Phys. Semesterber. **1** (1949), 77–88; Math. Ann. **126** (1953), 79–92; MR **13**, 767; **15**, 197], E. Sperner [Arch. Math. **5** (1954), 458–468; MR **16**, 278] und A. Karzel [ibid. **6**, 66–76 (1954), 284–295 (1955); MR **16**, 395; **17**, 776] die Einführung projektiv erweiterter Gruppenräume für alle Fälle der absoluten Geometrie zu geben, was in den bisherigen Untersuchungen noch nicht erreicht war, und daneben die einheitliche Begründung der absoluten Geometrien einschl. der drei Ausartungsfälle durch Gabelung aus 2 gemeinsamen Axiomen. Alle Resultate sind ausführlich abgeleitet, und die Arbeit ist unabhängig von den zitierten Literaturstellen verständlich. — Ausgangspunkt ist eine von einer Menge E involutorischer Elemente $a, b, \cdots, x, y, \cdots$ erzeugte abstrakte Gruppe G mit den allgemeinen Elementen $\alpha, \beta, \cdots$; die Menge aller involutorischen Elemente von G heiße J. Für $\alpha \neq 1$ heißt die Menge der $x \in E$ mit $\alpha x \in J$ das Büschel $[\alpha]$. Ein Büschel B heißt Lotebüschel, wenn es ein x gibt mit $B=[x]$. B heißt eigentliches Büschel, wenn es $xy \in J$ gibt mit $B=[xy]$. Wenn $\alpha \neq 1$ und $[\alpha]$ ein Lotebüschel ist, das nicht eigentliches Büschel ist, so erzeugt α ein uneigentliches Lotebüschel. Die Gruppenaxiome lauten nun: I (Axiom für die Büschelung): Aus $x, y, z \in [\alpha]$ folgt $xyz \in E$. II (Axiom für die Existenz von Verbindungselementen): Falls α kein uneigentliches Lotebüschel erzeugt, gibt es zu α, β ein x mit $\alpha x, \beta x \in J$ oder $\alpha x \in J$, $[\beta]=[x]$. — Der Fall, daß stets $abc \in E$ ist, ist geometrisch trivial. Es bleibt also als Axiom III: Es gibt u, v, w mit $uvw \notin E$. Aus I–III ergeben sich detailliertere Aussagen über die Gruppe G. — Der Fall (Axiom IV), daß E kein Zentrumselement von G enthält, ist im Gegensatz zu den beiden sonst noch möglichen Fällen allein nicht trivial, so daß künftig noch Axiom IV benutzt wird. Die Fallunterscheidung zwischen Euklidischer und Nichteuklidischer Geometrie wird jetzt gegeben durch Axiom E: Es gibt $u \neq v$ mit $[u]=[v]$, resp. Axiom N: Aus $[a]=[b]$ folgt $a=b$. — Im Euklidischen und Nichteuklidischen Fall wird nun ein Gruppenraum mit Hilfe der Elemente von G (Axiome I–IV und E resp. N) definiert, wobei im Euklidischen Fall uneigentliche Punkte mit Hilfe der Elemente der Faktorgruppe der Bewegungsgruppe nach der Translationsgruppe einzuführen sind, im hyperbolischen Fall Paare singulärer Büschel (ein Büschel U heißt singulär, wenn es kein $\xi \in J$ gibt mit $U=[\xi]$).

Der Gruppenraum erweist sich dann im Euklidischen und Nichteuklidischen Fall als ein projektiver Raum. — Darüber hinaus zeigen die projektiven Gruppenräume die Struktur der Bewegungsgruppe auf, nämlich ihre Darstellbarkeit durch Biquaternionen im Euklidischen Fall, durch Quaternionen im elliptischen Fall, in dem keine singulären Büschel existieren, und im hyperbolischen Fall durch eine volle Matrixalgebra 2. Grades über dem Koordinatenkörper. Ferner läßt sich nach Einführung des kommutativen Koordinatenkörpers auch die metrische Grundform gewinnen, womit die Algebraisierung der absoluten Geometrie abgeschlossen ist. Nach Abspaltung der erwähnten trivialen Ausartungsfälle kennzeichnen die Axiome I und II genau die absolute Geometrie im Sinne einer maximalen Geometrie, d.h. mit maximalem Eigentlichkeitsbereich in ihrer projektiven Erweiterung. *R. Moufang* (Frankfurt a.M.).

Ewald, Günther. Begründung der Geometrie der ebenen Schnitte einer Semiquadrik. Arch. Math. **8** (1957), 203–208.

Verf. konstruiert ein Modell eines dreidimensionalen projektiven Raumes, in dem die Ebenen, Geraden und Punkte resp. als Kreise, Büschel und Bündel einer durch 4 Axiome definierten verallgemeinerten Kreisgeometrie erklärt sind, die nur die Grundbegriffe "Kreis" und "orthogonal" enthalten. In diesem projektiven Raum lässt sich dann eine Polarität erklären, die wegen des vierten der Axiomen kein Nullsystem ist und sich analytisch darstellen lässt nach einem Resultat von R. Baer [Linear algebra and projective geometry, Academic Press, New York, 1952; MR **14**, 675], und deren selbstkonjugierte Punkte eine Semiquadrik bilden. Die Punkte der Semiquadrik lassen sich dann eineindeutig den zu sich selbst orthogonalen Kreisen zuordnen, und die Punkte eines ebenen Schnittes der Semiquadrik entsprechen genau den Kreisen, die zu einem festen Kreis orthogonal sind. Die Fälle, dass die Semiquadrik nullteilig oder schwach convex oder convex oder eine Regelfläche ist, werden durch ein Zusatzaxiom gekennzeichnet, womit die vollständige axiomatische Kennzeichnung aller Typen der hier betrachteten allgemeinen Kreisgeometrie geleistet ist. *R. Moufang* (Frankfurt a.M.).

Pickert, Günter. Zur Grundlegung der analytischen Geometrie. Jber. Deutsch. Math. Verein. **60** (1958), Abt. 1, 97–100.

Citing W. B. Jurkat [same Jber. **59** (1957), Abt. 1, 87–92; MR **19**, 54] and E. Artin [Rep. Math. Colloq. (2) **2** (1940), 15–20; MR **3**, 179], the author proposes an axiomatic foundation in 5 axioms which take the "double direction" as fundamental concept; he defines the dilatation as an endomorphism of the vector group leaving each double direction invariant. The proofs are sketched and will be published as appendix II in the third edition of his "Analytische Geometrie" [2. Aufl., Geest-Portig, Leipzig, 1955; for a review of the 1st ed. see MR **15**, 339]. Seven axioms on "direction" are stated for an analytic geometry with ordered division ring. *S. R. Struik.*

Aeppli, A. Das Taktionsproblem von Apollonius, angewandt auf die vier Berührungskreise eines Dreiecks. Elem. Math. **13** (1958), 25–30.

The problem of Apollonius to construct the eight circles touching three given ones is here solved for the

special case where the three circles are any triplet chosen from the inscribed and escribed circles of a triangle. As the sides of the triangle and the Feuerbach nine-point circle are solutions, the problem reduces to the construction of four circles. Formulae are given expressing the radii of the 16 circles so obtained in terms of the sides of the triangle and the radius of the Feuerbach circle.

F. A. Behrend (Melbourne).

Rosati, Luigi Antonio. Sui piani desarguesiani affini "non-ciclici". Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **22** (1957), 443–449.

A finite Desarguesian plane of order $n=p^r$ has a collineation group G of order n^2-1 transitive and regular on the finite points excluding the origin. There is always a cyclic group with this property. Hughes [Amer. J. Math. **78** (1956), 650–674; MR **18**, 921] raised the question as to the existence of finite planes with a non-cyclic group of this kind. This question is answered in this paper. A Desarguesian plane of order n has a non-cyclic group of this kind if and only if there is a near-field of order n^2 whose center is of order n. *Marshall Hall, Jr.*

Pajares Diaz, E. On a transcendental curve. Las Ciencias **22** (1957), 637–650. (Spanish)

The curve is the following generalization of the logarithmic spiral. In a plane π let O, P_0 be points, Δ a ray issuing from P_0 perpendicular to OP_0. A point P traverses Δ while a plane π' coincident with π rotates about O at a rate proportional to $\log P_0P$; the curve is traced out on π' by P.

Amici, Andrea. Sulle ellissi di Steiner e la geometria del triangolo. Ricerca, Napoli (2) **8** (1957), Luglio-Dicembre, 32–38.

Klingenberg, Wilhelm. Affine Ebenen mit Orthogonalität. Arch. Math. **8** (1957), 199–202.

This paper gives a simple proof of a result of Naumann [Math. Ann. **131** (1956), 17–27; MR **18**, 61] on affine planes with a non-trivial orthogonality relation which also satisfy the axiom: If P, P' and Q, Q' are pairs of distinct points on two orthogonal lines, then there is a collineation taking P into P', Q into Q' and every line into a line orthogonal to itself. This axiom implies that the plane can be coordinatized by a division ring D which possesses an automorphism α: $u\to\bar{u}$ whose square is an inner automorphism of D. Orthogonality can be expressed in terms of this automorphism.

Marshall Hall, Jr. (Columbus, Ohio).

Al-Dhahir, M. W. On the Pappus proposition and its configuration. Bull. Coll. Arts Sci. Baghdad **2** (1957), 96–98.

As an example that "it is possible to prove a special, but non-degenerate, form of the Pappus theorem without the use of the commutativity of multiplication or any of its equivalents" [cf. H. F. Baker, Principles of geometry, v. I, Cambridge, 1922, p. 56], the author proves the following proposition: If any two of three triangles forming a Pappus configuration are polar under a certain polarity, the third triangle is self-polar. *N. A. Court.*

Fulton, Curtis M.; and Stein, Sherman K. The passage from geometry to algebra. Math. Ann. **134** (1957), 140–142.

This paper defines addition and multiplication for points of a conic C in the real projective plane. Let ∞ and 0 be two distinct points of C, separating C into a positive part P and a negative part N. Let a and b be two points of $C-\infty$. Let the line joining a and b (tangent to C if $a=b$) meet the tangent through ∞ in a point U. Let the second tangent to C through U meet C at c. Then put $c=\frac{1}{2}(a+b)$. Similarly let the line through a and b (both on P) meet the line through ∞ and 0 at V. Let a tangent to C through V meet P at d. Then put $d=ab$. Write either binary operation $\frac{1}{2}(a+b)$ or ab as $a\circ b$. Then the idempotent law $a\circ a=a$ and the commutative law $a\circ b=b\circ a$ both hold. Theorems on conics quoted from Veblen and Young's "Projective geometry" [v. 1, Ginn, Boston, 1910] prove the "medial" law $((a\circ b)\circ(c\circ d))=((a\circ c)\circ(b\circ d))$. From this the operation $\cdot$ defined by $(b\circ x)\cdot(b\circ y)=x\circ y$ defines an Abelian group, respectively addition or multiplication.

{These definitions are supposed to have the virtue of avoiding the choice of an arbitrary point $\neq 0, \infty$ on C as the unit point 1. But the use of the fixed element b in the definition of the dot operation appears quite as arbitrary to the reviewer. From an axiomatic standpoint, the use of a "positive" part of the conic is undesirably restrictive.}

Marshall Hall, Jr. (Columbus, Ohio).

Weiner, L. M. The theorem of Pythagoras in n dimensions. Univ. Nac. Tucumán. Rev. Ser. A. **11** (1957), 28–31.

One possible generalization of a right-angled triangle is an n-dimensional simplex having n mutually orthogonal edges at one vertex. The cell opposite to this vertex is naturally called the "hypotenuse". Donchian and Coxeter [Math. Gaz. **19** (1935), 206] proved that the square of the content of the hypotenuse is equal to the sum of the squares of the contents of the n remaining cells. The author expresses this result as a property of determinants.

H. S. M. Coxeter (Toronto, Ont.).

★ **Rehbock, Fritz. Darstellende Geometrie.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957. xv+232 pp. DM 23.00.

This is a text on technical descriptive geometry, intended for students of mathematics, engineering and architecture. The emphasis is on the practice of drawing: there are 110 excellent full-page figures with explanatory text conveniently arranged on the adjacent pages; no attempt is made to present a systematic theoretical background. *F. A. Behrend* (Melbourne).

★ **Strubecker, Karl. Vorlesungen über darstellende Geometrie.** Vandenhoeck & Ruprecht, Göttingen, 1958. x+324 pp. DM 16.80.

Except for minor additions, this book represents a course of lectures on Descriptive Geometry given by the author since 1953/54 to students of mathematics, physics, engineering and architecture. The main aim of the book is to provide a coherent basis for the understanding of the topic (the actual technique of drawing being delegated to accompanying practice classes). Every opportunity is used to touch on relevant theoretical background and to branch out into related fields; the following examples are typical: Desargues's theorem, affine transformations, the fundamental theorem of algebra and Bézout's theorem, tangent, normal and binormal of twisted curves, ruled surfaces, curvature of surfaces, Meusnier's, Euler's and Dupin's theorems. Apart from its main theme, the book is thus a natural introduction to the simplest problems

and methods of analytic, affine, projective, algebraic and differential geometry, and its spirit may well be characterized by Gauss's remark (quoted on p. 2) that descriptive geometry is "a nourishing food for the enlivening and sustenance of the true geometrical spirit (sometimes lacking in the mathematics of the moderns)". — Table of contents: I. Mapping methods. II. Parallel projection and perspective affinity. III. The top and front view method. IV. Construction problems using front and top views. V. Representation of polyhedra by top and front views. VI. Oblique views. VII. Right circular cylinder. VIII. Right circular cone. IX. Skew circular cylinder and cone. X. Sphere. XI. Algebraic curves and surfaces. XII. Surfaces of revolution. XIII. Plane sections of surfaces of revolution. XIV. Intersection of two surfaces of revolution. XV. Intersection of two arbitrary surfaces. XVI. Helices and screw surfaces. XVII. Supplements on the theory of curvature of surfaces. *F. A. Behrend.*

Convex Domains, Integral Geometry

Hadwiger, H.; und Debrunner, H. Über eine Variante zum Hellyschen Satz. Arch. Math. 8 (1957), 309–313.

Let p, q, n, and k be positive integers, with $p \geqq q \geqq 2$. A family of sets is said to have the (p, q)-property provided it is true that among each p sets of the family some q have a common point. An n-separation of the family is a partition of it into n subfamilies, each with nonempty intersection. Let $N(p, q, k)$ be the smallest integer such that whenever a finite family of convex sets in E^k has the (p, q)-property, then it admits an n-separation with $n \leq N(p, q, k)$. (If no such integer exists, set $N(p, q, k) = \infty$.) It is easily verified that $N(p, q, k) = \infty$ when $q \leqq k$, and the authors observe that always $N(p, q, k) \geqq p - q + 1$. The well-known theorem of Helly asserts that $N(k+1, k+1, k) = 1$, and this is extended by the principal result of the present paper, which is that $N(p, q, k) = p - q + 1$ whenever $p \geqq q \geqq k+1$ and $kq \geqq (k-1)p + k + 1$. (The proof is by induction on $p+q$ for a fixed dimension k.) In particular, $N(p, q, 1) = p - q + 1$ for $p \geqq q \geqq 2$, and $N(p, q, 2) = p - q + 1$ for $2q - 3 \geqq p \geqq q \geqq 3$. The story is incomplete even in the two-dimensional case, and in particular the value of $N(4, 3, 2)$ is unknown. (The authors refer to an example of L. Danzer showing that $N(4, 3, 2) \geqq 3$.) *V. L. Klee, Jr.*

Ohmann, D. Ein allgemeines Extremalproblem für konvexe Körper. Monatsh. Math. 62 (1958), 97–107.

The author denotes the integral of a direction function $g(\zeta)$ over the surface of a convex body K in Euclidean n dimensional space by $J(g; K)$. He solves the problem of finding that convex body of largest volume for which $J(g_i; K)$ $(i = 1, 2, \cdots, k)$ have prescribed values or satisfy prescribed inequalities. The result is closely connected with Minkowski's inequality concerning mixed volumes. A large number of applications are given and it is shown that many known results can be generalised by using this method. *H. G. Eggleston* (London).

★ **Alexandrow, A. D. Konvexe polyeder.** Mathematische Lehrbücher und Monographien. Herausgegeben von der Deutschen Akademie der Wissenschaften zu Berlin, Forschungsinstitut für Mathematik. II. Abteilung: Mathematische Monographien, Bd. VIII. Akademie-Verlag, Berlin, 1958. x+419 pp. DM 37.00.

A translation by H. Abetz from the Russian original reviewed in MR 12, 732.

Stein, Sherman K. Continuous choice functions and convexity. Fund. Math. 45 (1958), 182–185.

A choice function defined over a family of sets is a function whose value for each member of the family is a point of the particular set concerned. The author shows that for an open subset of real n-dimensional Euclidean space to be convex it is necessary and sufficient that there exist a continuous choice function defined over its family of $(n-1)$-dimensional sections. Other results and some open questions are also given. The methods depend upon determining the degree of appropriately defined mappings and using the barycentric properties of convexity.

H. G. Eggleston (Cambridge, England).

Lippmann, Horst. Eine analytische Charakterisierung der Sinusfunktion mit Anwendungen auf die Winkelgeometrie in metrischen Raumen. Univ. e Politec. Torino. Rend. Sem. Mat. 16 (1956–57), 227–268.

The characterization of the sine function referred to in the title is obtained by establishing the following result: Let $h(x)$ be a real function that is (a) defined almost everywhere in the interval $(0, \pi)$ and summable there, and (b) satisfies for almost all x, y in $(0, \pi)$ the addition law $h(x \pm y) = \frac{1}{2}[h(x)h(\pi - y) \pm h(y)h(\pi - x)]$, where all arguments are in $(0, \pi)$. Then $h(x) = 2 \sin[(2k+1)x/2]$ almost everywhere in $(0, \pi)$.

A real function $w(p_1; p_2, p_3)$ defines an angle-function in a metric space M provided it is defined for every triple p_1, p_2, p_3 of M, $p_2 \neq p_1 \neq p_3$, and satisfies the following conditions: (a) $w(p_1; p_2, p_3) = 0$ for $p_2 = p_3$ or $p_1 p_2 p_3$ or $p_1 p_3 p_2$, and $w(p_1; p_2, p_3) > 0$ otherwise; (b) $w(p_1; p_2, p_3) = w(p_1; p_3, p_2)$; (c) $w(p_1; p_2, p_3) = \pi$ if $p_2 p_1 p_3$. Here $p_i p_j p_k$ denotes the relation $p_i p_j + p_j p_k = p_i p_k$, $p_i \neq p_j \neq p_k$. The function $w_e(p_1; p_2, p_3)$ defined by $\cos w_e(p_1; p_2, p_3) = (p_1 p_2^2 + p_1 p_3^2 - p_2 p_3^2)/2 p_1 p_2 \cdot p_1 p_3$, $0 \leqq w_e \leqq \pi$, satisfies (a), (b), (c). The principal result of the paper: Let w be an angle-function defined in a complete, metrically convex, metric space, and suppose (i) the sum of two adjacent angles is π, (ii) the angle-sum of each triangle is π and (iii) similar triangles have corresponding angles equal. Then $w = w_e$, and M is congruently imbeddable in generalized euclidean space. The lengthy but interesting proof entails a systematic development of the angle geometry of M.

L. M. Blumenthal (Columbia, Mo.).

Stamey, W. L. On generalized euclidean and non-euclidean spaces. Pacific J. Math. 7 (1957), 1505–1511.

Denote the distance of two elements x, y of a metric space M by xy. M is externally convex provided it contains for each pair of elements p, q an element r such that $pq + qr = pr$, $p \neq q \neq r$. The author considers the following localization of external convexity: if $S_{p,q}$ denotes a metric segment with endpoints p, q, there exists a positive number $\delta(p)$ such that $s \in S_{p,q}$, $0 < ps < \delta(p)$ implies the existence of an element t of M such that $sp + pt = st$, $s \neq t$. The principal result obtained is that if M is complete and convex, and has, moreover, the property that for each two distinct elements p, r it contains an element q such that (i) $pq + qr = pr$, $p \neq q \neq r$, and (ii) each quadruple of its elements containing p, q, r is congruently imbeddable in the plane, then the localized property implies the (global) external convexity of M. It follows at once that certain theorems of Day [Bull. Amer. Math. Soc. 62 (1956), 45–46] and the reviewer [Pacific J. Math. 5 (1955), 161–167; MR 16, 1139] that characterize metrically generalized euclidean space are valid when the kind of local external convexity defined above is substituted for

the condition of external convexity used in those theorems. *L. M. Blumenthal* (Columbia, Mo.).

See also: Topological Vector Spaces: Björck.

Differential Geometry

Muracchini, Luigi. Osservazioni sull'applicabilità proiettiva di due trasformazioni puntuali fra piani. Boll. Un. Mat. Ital. (3) **12** (1957), 176–182.

On doit à L. Muracchini et M. Villa l'importante notion de l'applicabilité projective de deux transformations analytiques de première espèce, T, T', entre deux plans [même Boll. (3), **7** (1952), 29–38; **10** (1955), 313–327; MR **14**, 205; **17**, 781]. Dans l'article présent l'auteur forme un système d'équations de Pfaff définissant les couples de transformations T, T' projectivement applicables. Il démontre, en se servant de méthodes de E. Cartan, que les couples de transformations T, T' projectivement applicables dépendent, en général, d'une fonction arbitraire de deux arguments. Une correspondance T n'est pas, par conséquent, en général, projectivement déformable. L'auteur donne la construction d'une classe de couples de correspondances T, T' projectivement applicables possédant la dite généralité. L'article contient en outre d'intéressantes remarques sur la généralité des couples de transformations T, T' projectivement applicables, formés avec la transformation T donnée. *O. Borůvka.*

Manifolds, Connections

Lichnerowicz, André. Sur la réductibilité des espaces homogènes riemanniens. C. R. Acad. Sci. Paris **243** (1956), 640–642.

Several theorems on homogeneous Riemannian manifolds are given, including (1) if a homogeneous Riemannian $V=G/H$ is simply connected and the group G is simple and non-compact with finite center, then V is irreducible, and 2) every homogeneous Kählerian G/H with G simple and compact is irreducible. *W. Ambrose.*

Kanitani, Jôyô. Sur la forme de Darboux relative une variété différentiable. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **30** (1957), 203–216.

The notion of the Darboux form of a hypersurface is generalized to an n-dimensional differentiable manifold, admitting a covariant differentiable tensor field $a_{ij}du^i\otimes du^j$ of the second order, by first introducing a canonical dominant projective connection and a covariant differentiable tensor field $b_{ijk}du^i\otimes du^j\otimes du^k$ of the third order. *C. C. Hsiung* (Bethlehem, Pa.).

See also: Partial Differential Equations: Peetre. **General Topology:** Balcerzyk and Mycielski.

Complex Manifolds

Remmert, Reinhold. Holomorphe und meromorphe Abbildungen komplexer Räume. Math. Ann. **133** (1957), 328–370.

Ceci constitue un exposé d'ensemble des problèmes et résultats concernant les applications holomorphes d'un espace analytique dans un autre. On y rappelle des notions et des théorèmes connus, et on fait aussi une nouvelle mise au point, avec démonstrations, des principaux résultats dus à l'auteur [Math. Ann. **130** (1956), 410–441; MR **19**, 170]. La définition d'un "espace analytique" adoptée ici équivaut à celle de H. Cartan, et aussi à celle de Behnke-Stein (en vertu d'un théorème récent de Grauert et Remmert); il s'agit donc d'espace analytique "normal", et non d'espace analytique au sens plus général de Serre [Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42; MR **18**, 511]. Il en résulte quelques complications techniques: chaque fois qu'intervient un sous-ensemble analytique M d'un espace analytique, on doit substituer à M l'espace normal associé M^* (cf. déf. 11). Une partie des résultats du présent mémoire serait valable pour les espaces analytiques au sens le plus général; toutefois, certains autres résultats supposent essentiellement qu'il s'agit d'espaces normaux.

On rappelle le théorème de Remmert-Stein (Satz 9), qui joue un rôle fondamental dans tous ces problèmes: si N est un sous-ensemble analytique de dimension $<d$ dans un espace analytique X, et si M est un sous-ensemble analytique purement d-dimensionnel dans $X-N$, alors l'adhérence $\bar{M}$ de M dans X est un sous-ensemble analytique purement d-dimensionnel dans X.

Soit $\tau:X\to Y$ une application holomorphe d'espaces analytiques; le "rang" $r_\tau(x)$ au point $x\in X$ est, par définition, la codimension, au point x, de la fibre $\tau^{-1}(\tau(x))$; le "rang global" $r_\tau(X)$ est $\sup_{x\in X} r_\tau(x)$. Dans le cas particulier où X et Y sont des variétés analytiques complexes, le rang $\rho_\tau(x)$ défini par la matrice jacobienne est $\leq r_\tau(x)$, et on a $\rho_\tau(x_0)=r_\tau(x_0)$ si $\rho_\tau(x)$ est constant pour x voisin de x_0. On démontre (Satz 17): l'ensemble des $x\in X$ tels que $r_\tau(x)\leq m$ (m entier donné) est un sous-ensemble analytique; en particulier, l'"ensemble de dégénérescence" E_τ (ensemble des $x\in X$ tels que $r_\tau(x)<r_\tau(X)$) est un ensemble analytique. Si $x\notin E_\tau$, il existe un voisinage U de x tel que $\tau(U)$ soit un sous-ensemble analytique au voisinage de $\tau(x)$, et sa dimension est égale à $r_\tau(x)$ (Satz 19). Un théorème fondamental est le suivant (Satz 23): si $\tau:X\to Y$ est holomorphe et propre (i.e.: l'image réciproque de tout compact est compacte), alors l'image $\tau(X)$ est un sous-ensemble analytique de Y, de dimension égale au rang $r_\tau(X)$. En particulier, si X est compact et si $\tau:X\to Y$ est holomorphe, $\tau(X)$ est un sous-ensemble analytique de Y; si en outre Y est un produit d'espaces projectifs, alors $\tau(X)$ est algébrique (th. de Chow).

Le Satz 23 ci-dessus est déduit d'un théorème plus général (Satz 22): soit $\tau:X\to Y$ une application holomorphe; s'il existe un compact $K\subset X$ et un ouvert $U\subset Y$ tels que chaque composante purement dimensionnelle de chaque fibre au-dessus de U soit rencontré par K, alors $\tau(X)\cap U$ est un sous-ensemble analytique de U. {Nota: le rapporteur n'est pas certain que ce théorème soit vrai (de toute manière, le Satz 23 est vrai); en effet, dans la démonstration, l'auteur admet implicitement que si X satisfait aux hypothèses de l'énoncé, chaque composante connexe de X y satisfait; or ceci est faux, mais cette fausseté n'infirme pas nécessairement le théorème lui-même.}

Si E_τ est vide, et si $\dim Y=r_\tau(X)$, alors τ est une application ouverte (Satz 28). Il y a une réciproque (Satz 29): si $\tau:X\to Y$ (où $\dim Y=r$) est holomorphe et ouverte, alors E_τ est vide, et $r_\tau(X)=r$. De plus (Satz 30): si τ est holomorphe, avec $r_\tau(X)=r$, $\dim Y=r$, et si chaque fibre possède au moins une composante connexe compacte, alors $\tau(E_\tau)$ se compose uniquement de points intérieurs à $\tau(X)$.

L'holomorphie d'une application $\tau:X\to Y$ (où X est irréductible) peut se caractériser à l'aide du graphe $G_\tau\subset X\times Y$ (mais il est essentiel de supposer que X et Y

sont des espaces analytiques normaux): pour que τ soit holomorphe, il faut et il suffit que G_τ soit un sous-ensemble analytique et que dim G_τ=dim X (Satz 31). On ne peut pas laisser tomber la condition dim G_τ=dim X; cependant elle est superflue si X et Y sont réunions dénombrables de compacts. Plusieurs applications de ce critère sont données.

Un dernier paragraphe est consacré à la notion d'"application méromorphe". Etant donnés des espaces analytiques X et Y (X irréductible), on appelle application méromorphe de X dans Y la donnée d'un sous-ensemble analytique irréductible $G_\tau \subset X \times Y$, de même dimension que X, tel que la projection $G_\tau \to X$ soit surjective, ait des fibres compactes, et que la fibre de x soit réduite à un point pour les x d'un ouvert dense dans X. Ceci généralise bien la notion d'application méromorphe dans un produit de droites projectives; on démontre que l'application $G_\tau \to X$ est propre, et définit G_τ comme "éclatement" de X ("eigentliche Modifikation"). Cet éclatement permet de remplacer τ par une véritable application holomorphe, et jouit d'une propriété universelle vis-à-vis des éclatements satisfaisant à cette condition. *H. Cartan* (Paris).

Lelong, Pierre. Ensembles singuliers impropres des fonctions plurisousharmoniques. J. Math. Pures Appl. (9) **36** (1957), 263–303.

Der Verf. betrachtet m-dimensionale differenzierbare Mannigfaltigkeiten V^m. $E \subset V^m$ ist eine abgeschlossene Teilmenge, so daß $\Omega = V^m - E$ zusammenhängend ist. Mit $L(V^m)$ bzw. $L(\Omega)$ wird eine Klasse von Funktionen über V^m bzw. Ω bezeichnet. Die Menge E heißt eine Menge hebbarer Singularitäten (ensemble singulier impropre) für $L(\Omega)$, wenn sich jede Funktion aus $L(\Omega)$ eindeutig zu einer Funktion aus $L(V^m)$ fortsetzen läßt. — Da die Probleme seiner Arbeit lokaler Natur sind, beschränkt sich der Verf. auf den Fall, daß V^m ein Teilgebeit D eines m-dimensionalen euklidischen Raumes ist. E ist eine Polmenge (ensemble R^m-polaire), d.h. eine Teilmenge von D, die lokal Minusunendlichkeitsmenge einer subharmonischen Funktion ist. D kann auch ein Teilgebeit des komplex n-dimensionalen komplexen Zahlenraumes $C^n \approx R^{2n}$ sein. Dann heißt $E \subset D$ eine komplexe Polmenge (ensemble C^n-polaire), wenn E lokal Minusunendlichkeitsmenge einer plurisubharmonischen Funktion ist. Insbesondere ist jede analytische Menge $A \neq D$ eine komplexe Polmenge, jede komplexe Polmenge eine (reelle) Polmenge. Das Verhalten von Polmengen bei holomorphen Abbildungen wird untersucht (Prop. 2), ferner der Durchschnitt von komplexen Polmengen mit analytischen Mengen (Prop. 3) und reellen Unterräumen (Cor. 3). Dabei wird eine an sich interessante Eigenschaft der plurisubharmonischen Funktionen hergeleitet: Jede plurisubharmonische Funktion ist auf den reellen Unterräumen $R^n \subset C^n$ lokal summierbar (Prop. 4). — Der Verf. gibt sodann eine einfache Charakterisierung der plurisubharmonischen Funktion an: (*) Eine subharmonische Funktion ist genau dann plurisubharmonisch, wenn sie durch jede komplex-lineare Abbildung wieder in eine subharmonische Funktion übergeführt wird. Dieser Satz wird angewendet, um von Eigenschaften subharmonischer Funktionen auf Eigenschaften plurisubharmonischer Funktionen zu schließen. Besonders wird er zur Herleitung von Fortsetzungssätzen benutzt. Zunächst wird gezeigt: Es sei $E \subset D$ eine abgeschlossene Polmenge, $V(x)$ ein subharmonische Funktion in $D-E$, die längs E gewissen Beschränktheitsbedingungen genügt. Dann läßt sich $V(x)$ nach angegebenen Vorschriften zu einer in ganz D subharmonischen Funktion fortsetzen (Prop. 7). Unter Verwendung von (*) ergeben sich daraus analoge Sätze für plurisubharmonische Funktionen (Th. 2) und auch für holomorphe Funktionen. Besonders interessant ist, daß man mit Hilfe der beschriebenen Methoden eine Verallgemeinerung des bekannten Radóschen Satzes gewinnen kann. Nach T. Radó ist jede stetige außerhalb ihrer Nullstellen holomorphe Funktion $w=f(z)$ überall holomorph. Der Verf. zeigt sogar: $w=f(z)$ ist überall holomorph, wenn $w=f(z)$ nur in $\{z \in D,\ f(z) \notin e_0\}$, e_0 eine harmonische Nullmenge, holomorph ist. Dieses Resultat wird noch allgemeiner ausgesprochen (Th. 3 und 4). — Am Ende der Arbeit sind "biharmonische Funktionen" behandelt. Es werden für diese Funktionen in bezug auf gewisse Mengen E Fortsetzungssätze hergeleitet, die von keinen Beschränktheitsbedingungen mehr abhängen (unter wesentlicher Verwendung des Begriffes "ensemble effilé", der in Abschnitt 4 untersucht wird). Da plurisubharmonische Funktionen insbesondere biharmonische Funktionen sind, folgen analoge Sätze für plurisubharmonische (und holomorphe) Funktionen. U.a. werden Klassen L_n von abgeschlossenen Mengen $E \subset D \subset C^n$ definiert. Jede Klasse L_n ist gegenüber abzählbarer Vereinigung abgeschlossen und enthält die mindestens 2-codimensionalen analytischen Mengen. Ferner gehört (unter einer zusätzlichen Voraussetzung) jede abgeschlossene Teilmenge $E' \subset E \in L_n$ wieder zu L_n. Der Verf. beweist abschließend folgenden sehr allgemeinen Satz: Die Mengen $E \in L_n$ sind Mengen hebbarer Singularitäten für die plurisubharmonischen und die holomorphen Funktionen.

H. Grauert (Princeton, N.J.).

Edwards, R. E. Algebras of holomorphic functions. Proc. London Math. Soc. (3) **7** (1957), 510–517.

Let X be a complex-analytic manifold and $H(X)$ the algebra of functions holomorphic on all of X. To what extent does $H(X)$ characterize X? It is clear that the class of manifolds considered must be suitably restricted, since for instance for all compact manifolds $H(X)$ is one-dimensional over the complex scalars. For the case when X has complex dimension 1, the problem has been settled by Bers, Kakutani and Chevalley, and Rudin [cf. S. Kakutani, Lectures on functions of a complex variable, Univ. of Michigan Press, 1955, pp. 71–83; MR **16**, 1125]. The author calls a manifold X (of any dimension) of type A if $H(X)$ separates points on X and provides local coordinates at each point of X, if also for each x in X the ideal $M(x)$ consisting of all functions in $H(X)$ vanishing at x is finitely generated, and if further every finite subset F of $H(X)$ without common zero generates all of $H(X)$ as ideal in $H(X)$. Theorem: Let X and Y be two manifolds of type A. Every algebraic isomorphism between $H(X)$ and $H(Y)$ is induced by an analytic isomorphism between X and Y. Theorem: Let X be a Stein manifold. Then X is of type A. Further, if the complex dimension of X is m, then every maximal ideal in $H(X)$ of codimension 1 admits a system of at most $2m+1$ generators and has the form $M(x)$ for some x. The proof of this theorem is obtained from sheaf theory as given in Séminaire H. Cartan, Ecole Norm. Sup., 1951–52 [MR **16**, 233], the proof of the statement about the number of generators being due to Cartan (unpublished). The author also considers certain generalizations of type A, and discusses the case of bicontinuous isomorphisms between $H(X)$ and $H(Y)$.

J. Wermer.

Algebraic Geometry

Spampinato, Nicolò. Su una superficie di ordine 9 e classe 9 dotata di un fascio lineare di cubiche autoduali. Ricerca, Napoli (2) 8 (1957), Luglio-Dicembre, 8–16.

Facciotti, G. L'iperbole equilatera sferica inviluppo. Period. Mat. (4) 35 (1957), 164–178.

The late G. Ascoli suggested to the author that the spherical quartic Q defined in a former paper on equilateral spherical hyperbolas, might be the envelope of the third sides of the family of equivalent spherical right triangles which have two sides coincident with the asymptotes of the equilateral spherical hyperbolas. The author derives the equations of the envelope Γ of the third sides of this family of spherical triangles and finds that Γ and Q are different spherical quartics. Hence Q is not the envelope suggested. *T. R. Hollcroft.*

★**Togliatti, Eugenio. Un'osservazione sulle reti omaloidiche di curve piane.** Scritti matematici in onore di Filippo Sibirani, pp. 281–284. Cesare Zuffi, Bologna, 1957.

Con riferimento ad una rete omaloidica di curve piane di ordine n con h punti base ordinari di molteplicità

$$r_1 \geqq r_2 \geqq \cdots \geqq r_h \geqq 1,$$

l'A. ricorda la disuguaglianza di Noether-Rosanes: $r_1+r_2+r_3>n$, e quella di Montesano (migliore della precedente, e valida per $n>2$): $r_1+r_2+r_5>n$. In questa nota l'A. migliora le disuguaglianze precedenti, dimostrando che per una rete con almeno 6 punti base (e quindi per $n>3$) è sempre $r_1+r_3+r_6>n$, e facendo vedere, con opportuni esempi, che non è possibile migliorare ulteriormente questa disuguaglianza. Sempre con esempi si dimostra pure che la somma $r_1+r_3+r_6$ può effettivamente raggiungere il minimo $n+1$.

M. Rosati (Roma).

Godeaux, Lucien. Remarques sur les couples de congruences stratifiables. Univ. e Politec. Torino. Rend. Sem. Mat. 16 (1956–57), 219–226.

Severi, Francesco. Alcune osservazioni sopra la caratterizzazione topologica delle superficie algebriche. Rend. Mat. e Appl. (5) 16 (1957), 161–169.

Italian version of the article on pp. 54–61 of "Studies in mathematics and mechanics, presented to Richard von Mises", Academic Press Inc., New York, 1954 [MR 16, 397].

★**van der Waerden, B. L. The invariant theory of linear sets on an algebraic variety.** Proceedings of the International Congress of Mathematicians, 1954, Amsterdam, vol. III, pp. 542–544. Erven P. Noordhoff N.V., Groningen; North-Holland Publishing Co., Amsterdam, 1956. $7.00.

Con riferimento alla teoria dei sistemi lineari di curve sopra una superficie algebrica, nell'ambito della geometria algebrica classica, l'A. ricorda che per giungere al concetto invariante di "sistema lineare completo" (full linear set) occorre sviluppare lo scioglimento delle singolarità e la teoria dei punti infinitamente vicini. In un suo precedente lavoro egli aveva esteso la teoria ad una dimensione n qualunque, sostituendo i punti infinitamente vicini con opportune valutazioni di dimensione $n-1$, ma presupponendo sempre lo scioglimento delle singolarità. Nella presente nota l'A. indica un procedimento, indipendente dallo scioglimento suddetto, che permette di costruire la teoria dei sistemi lineari basandosi soltanto su enti e nozioni invarianti, naturalmente per una varietà algebrica definita sopra un campo qualunque. La nota si chiude con alcuni esempi, dove sono indicati anche alcuni problemi aperti. *M. Rosati* (Roma).

Kawahara, Yûsaku. On the theory of differential forms on algebraic varieties. Nagoya Math. J. 11 (1957), 13–39.

Suppose that V is an affine algebraic variety given by the generic point $x_1, \cdots, x_{n+1}$ over the field k and the corresponding defining polynomial $F(X_1, \cdots, X_{n+1})=0$, so that x_{n+1} is separable over $k(x_1, \cdots, x_n)$. The author extends the methods of valuation theory used for the proof of the Jacobian formula for the differential dx in the case of one variable so as to obtain a characterization [as in Picard's classical work] of the subadjoint polynomials $\Phi(X_1, \cdots, X_{n+1})$ so that $\Phi(X_1, \cdots, X_{n+1})F'(X_1, \cdots, X_{n+1})^{-2}dx_1 \cdots dx_n$ is finite at every simple point of the derived normal variety [integral closure] V^* belonging to V: $\Phi(X_1, \cdots, X_{n+1})$ must lie in the conductor of the coordinate ring of V in that of V^*. Based on this result (Theorem 3), generalizations to projective varieties and differentials of degree less than n are made. Furthermore, restricting himself to characteristic 0, the author proves that linearly independent closed differential forms which are finite at every simple point of a projective variety of dimension $\geqq 2$ have linearly independent restrictions on every generic hyperplane section. The proof makes use of a reduction to dimension 2 and rests on a generalization of Castelnuovo's Lemma stating that $Af_{x'}+Bf_{y'}+Cf_{z'}=0$, with forms A, B, C of degree $<m-1$ and $f(x, y, z)=0$ an irreducible curve of degree m without singularities except nodes, can only hold if A, B, C vanish.

O. F. G. Schilling (Chicago, Ill.).

Borel, Armand. Groupes linéaires algébriques. Ann. of Math. (2) 64 (1956), 20–82.

This paper contains a comprehensive theory of linear algebraic groups G over an algebraically closed field of arbitrary characteristic. The author first studies diagonalizable groups, i.e. groups whose elements may be simultaneously reduced to diagonal form. Such a group may be defined by monomial relations between the diagonal coefficients of its matrices (these matrices being assumed to be diagonal). A diagonalizable group which is connected is called a torus; torus groups have the important property that their rational representations are all semi-simple.

If x is any invertible matrix, the smallest algebraic group containing x is called the adherent group of x; the author shows that this group always contains the semi-simple and the unipotent component x_s and x_u of x (x_s is semi-simple, x_u is unipotent, which means that $1-x_u$ is nilpotent, and $x=x_sx_u=x_ux_s$). As an application, it is shown that every commutative linear algebraic group is the product of a diagonalizable group by a group of unipotent elements. Moreover, it is also established that rational representations of a linear algebraic group map semi-simple (resp. unipotent) elements upon elements of the same kind.

The author then studies the case where G is solvable. He gives a new proof of the theorem of Lie-Kolchin to the effect that a connected solvable linear algebraic group G may be reduced to triangular form. Moreover, the uni-

potent elements of G form a normal subgroup G_u and, if T is a maximal torus of G, G is the semi-direct product of G_u and T (i.e. the mapping $(s, t)\to st$ is an isomorphism of the variety $G_u\times T$ with the variety G). The maximal tori of G are all conjugate to each other in G; if G is nilpotent, it has only one maximal torus T which is the set of semi-simple elements of G and which is in the center of G.

Consider now the general case where G is an arbitrary connected algebraic linear group. The main idea of the author to study this case is to make use of those homogeneous spaces G/H of G which are complete varieties. It is proved that, if a connected solvable group S operates on a complete variety V, then V must contain at least one fixed point for S. On the other hand, a direct construction shows that there always exists a solvable connected subgroup B of G such that G/B is complete (and even projective); B is then maximal in the class of connected solvable subgroups of G. The reviewer has proposed to call Borel subgroups of G the maximal solvable connected subgroups of G; using this terminology, it is established that all Borel subgroups of G are conjugate to each other in G; this theorem had already been stated for Lie groups by Morosow [C. R. Dokl. Acad. Sci. URSS (N.S.) **36** (1942), 83–86; MR **4**, 187], but Morosow's proof, based on computations in the Lie algebra, was not entirely correct. Making use of this theorem and of previously established results on solvable groups, the author shows that all maximal tori of G are conjugate to each other in G; if T is any one of them, then the centralizer of T is a nilpotent group which is contained in every Borel group which contains T; every semi-simple element is contained in some maximal torus. A group whose elements are all unipotent is nilpotent. The author gives new proofs for Kolchin's results on quasi-compact and anticompact groups.

The last section of the paper is concerned with the theory of Cartan subgroups of a connected algebraic group G (i.e. of maximal nilpotent subgroups C of G with the property that every subgroup of finite index of C is of finite index in its normalizer in G). It is proved that the Cartan subgroups are the centralizers of the maximal tori. For any x in G, denote by x_s the semi-simple component of x; if the centralizer Z of x_s is of smallest possible dimension, then x is called regular; Z is then a Cartan subgroup and is the only Cartan subgroup containing x; conversely, any element which belongs to only one Cartan subgroup is regular. The set of regular elements contains a non-empty open subset of G (in the Zariski topology).

Let f be a rational representation of G; then f maps the Borel subgroups (resp. the maximal tori, the Cartan subgroups, the regular elements) of G upon Borel subgroups (resp. maximal tori, Cartan subgroups, regular elements) of $f(G)$. The reciprocal image of a Borel subgroup (resp. a maximal torus, a Cartan subgroup) of $f(G)$ contains a Borel subgroup (resp. a maximal torus, a Cartan subgroup) of G.

{Since this paper has appeared, the reviewer has established (but not yet published) the following results, which complete those of the author: any Borel subgroup of G is its own normalizer in G; if C is a Cartan subgroup of G, the unipotent elements of C lie in the radical of G (i.e. in the largest connected solvable normal subgroup of G).} *C. Chevalley* (Paris).

See also: Special Functions: Rankin. **Lie Groups and Algebras:** Mostow.

NUMERICAL ANALYSIS

Numerical Methods

Glushkov, V. M. On certain problems of computation technique and mathematical problems connected with them. Ukrain. Mat. Ž. **9** (1957), 369–376. (Russian. English summary)

Campbell, Lloyd. Tchebyshev polynomial approximation of real functions. Ordnance Comput. Res. Rep. **5** (1958), no. 2, 14–18. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D.C.)

The author describes a procedure which has been programmed for the ORDVAC and EDVAC computing machines, for finding polynomial approximations to a function defined by a power series. The approximations are designed to give the same accuracy as the Taylor series expansion, but with fewer terms. Starting with the original power series, the coefficients of expansion in a series of Tchebyshev polynomials are obtained. The new series is then truncated in such a way that the desired accuracy is obtained. The resulting sum is then rewritten in powers of x. Application of this method for certain cases resulted in savings of from 1 to 15 terms over that required with Taylor series. The procedure is described in the book by Lanczos [Applied analysis, Prentice Hall, Englewood Cliffs, N.J., 1956; MR **18**, 823].

{It should be pointed out that there are now computer programs available which are designed to obtain an approximate optimum polynomial fit of given degree in the minimax sense, to a given function in an interval, the function not necessarily being given as a power series.

In the author's statement, "Tchebyshev proved that the maximum error in an expansion of a power series into Tchebyshev polynomials is less than the maximum error of an expansion into any other set of ultraspherical polynomials," the word "ultraspherical" should be carefully noted. If more general polynomial expansions are considered, then the Tchebyshev expansion may not give the smallest error, although in many practical cases it may be nearly optimum.} *D. M. Young, Jr.*

Heurtaux, Jacqueline. Tables de polynomes d'interpolation avec seulement deux abscisses distinctes. Chiffres **1** (1958), 25–34.

This paper gives polynomial interpolation formulae agreeing with a given function $f(x)$, and its derivatives $f'(x)$, $f''(x)$, $\cdots$, $f^{(n-1)}(x)$, at each of two arguments $x=a$, $x=b$; the polynomial is of degree $(2n-1)$. Formulae are given for $n=2$, 3, 4, and tables of coefficients $Q_p{}^n(x)$, $R_p{}^n(x)$ in the formulae, namely

$$P_{2n-1}(x)=\sum_{p=0}^{n-1}\{Q_p{}^n(x)f^{(p)}(0)+R_p{}^n(x)f^{(p)}(1)\},$$

where $P_{2n-1}(x)$ is the polynomial approximation to $f(x)$.

Formulae and tables are for the standard range $a=0$, $b=1$, and tables are for $x=0(.01)1$; values are exact for $n=2,3$, and to 10 decimals, $p=0$, 1, 2, and 11 decimals, $p=3$, when $n=4$.

The error term is discussed and a practical example given. *J. C. P. Miller* (Cambridge, England).

Wolf, H. Beitrag zur Ausgleichung von untereinander abhängigen Beobachtungen. Z. Vermessungswesen 83 (1958), 113–120.

J. M. Tienstra [Theory of the Adjustment of Normally Distributed Observations, Argus, Amsterdam, 1956] has treated the problem of the adjustment of observations in the case when the observations are not independent, as usually assumed, but are correlated. Tienstra employed tensors. Wolf here provides computational schemes for the execution of the Tienstra adjustments, using the Gaussian notation on the lines of W. Grossmann [Grundzüge der Ausgleichungsrechnung, Springer, 1953; MR 15, 650] and remarking that despite the elegance and brevity of the tensor methods, they are not much use in computation.

He treats the observed quantities as if they, with their weights and correlations, had resulted from two or more previous adjustments on independent observations. He provides computer directions for adjustments by observation equations and by condition equations, and for the determination of the errors. He discusses the problem of finding the weights and correlations in the physical situation where they are not determined from previous adjustments. *J. A. O'Keefe* (Chevy Chase, Md.).

Ascher, Marcia; and Forsythe, George E. SWAC experiments on the use of orthogonal polynomials for data fitting. J. Assoc. Comput. Mach. 5 (1958), 9–21.

Let $\xi=(x_1, \cdots, x_m)$ be a set of real numbers, and let $f(x_1), \cdots, f(x_m)$ be the observed or computed values of some function of x at the points of ξ. Let $f_n^{(\xi)}(x)$ be the unique polynomial of degree n which best fits the data values $f(x_i)$ in the least-squares sense, i.e., the sum $S(q_n)=\sum_{i=1}^m \{f(x_i)-q_n(x_i)\}^2$, where q_n is a polynomial of degree n, is minimized by $q_n(x)=f_n^{(\xi)}(x)$. Forsythe [J. Soc. Indust. Appl. Math. 5 (1957), 74–88; MR 19, 1079] discussed the problem of finding $f_n^{(\xi)}(x)$ on a digital computer and outlined certain advantages of determining $f_n^{(\xi)}(x)$ in terms of polynomials $p_k(x)$ orthogonal over ξ. It was further proposed that the $p_k(x)$ be computed from their three-term recurrence

$$\text{(A)} \qquad p_{k+1}(x)=(x-\alpha_{k+1})p_k(x)+\beta_k p_{k-1}(x),$$

where $p_0(x)=1$, $\beta_0=0$, $\alpha_{k+1}=\sum_{i=1}^m x_i\{p_k(x_i)\}^2/\sum_{i=1}^m\{p_k(x_i)\}^2$, $\beta_k=\sum_{i=1}^m \{p_k(x_i)\}^2/\sum_{i=1}^m \{p_{k-1}(x_i)\}^2$. The $p_k(x)$ satisfy the orthogonality relation $\sum_{i=1}^m p_h(x_i)p_k(x_i)=0$, $h\neq k$. The Fourier coefficients t_k of $f(x_i)$ are defined by $t_k=\sum_{i=1}^m f(x_i)p_k(x_i)/\sum_{i=1}^m \{p_k(x_i)\}^2$. The desired least-squares polynomial is $f_n^{(\xi)}(x)=\sum_{k=0}^n t_k p_k(x)$. In this paper, the authors describe their experiment with the consistent use of the recurrence (A) for the data-fitting problem on the automatic digital computer SWAC, and a summary of typical results is presented. *E. Frank* (Chicago, Ill.).

Kogbetliantz, E. G. Computation of arcsin N for $0<N<1$ using an electronic computer. IBM J. Res. Develop. 2 (1958), 218–222.

Useful methods are derived for computing arcsin N to a given accuracy with as few multiplications and divisions as possible. The methods are based on Chebyshev expansions and rational approximations derived from them. As with other methods, they should be used with care, since arcsin N is singular at $N=\pm 1$.

C. B. Haselgrove (Manchester).

v. Gorup, G. Formeln und Tabellen für ein den Hankel'schen Zylinderfunktionen verwandtes Integral. Mitt. Max-Planck-Inst. Strömungsf. **15** (1957), 58 pp.

This paper gives a study of the functions

$$T_n(z)=\int_0^\infty e^{-nt}e^{-z\cosh t}dt,$$

$$S_n(z)=e^z T_n(z)=e^z K_n(z)-R_n(z),$$

where $K_n(z)$ is the Bessel function and $R_n(z)$ is a polynomial of degree n in $1/z$. Tables are given only for the case $z=-ik$, a pure imaginary. Then the author writes

$$S_n(-ik)=\alpha_n(k)+i\beta_n(k)=\int_0^\infty e^{-nt}e^{ik(\cosh t-1)}dt,$$

where α_n, β_n satisfy

$$\alpha_{n+1}=\alpha_{n-1}-\frac{2}{k}n\beta_n$$

$$\beta_{n+1}=\beta_{n-1}-\frac{2}{k}(1-n\alpha_n),$$

with

$$\alpha_0+i\beta_0=\tfrac{1}{2}\pi e^{ik}\{-Y_0(k)+iJ_0(k)\},$$

$$\alpha_1+i\beta_1=\tfrac{1}{2}\pi e^{ik}\{-J_1(k)-iY_1(k)\}-\frac{i}{k},$$

where J_0, Y_0, J_1, Y_1 are Bessel functions. The tables give 8-decimal values of Re T_n, Im T_n, α_n, β_n, $n^{-1}-\alpha_n$ for $n=0(1)40$ and $k=.001$, .002, .005, .01(.01).06(.02).1(.1)2(1).7 A description of the calculation is given.

J. C. P. Miller (Cambridge, England).

★ **Riley, Vera; and Gass, Saul I. Linear programming and associated techniques.** Bibliographic Reference Series No. 5. Published for Operations Research Office, The Johns Hopkins University, by The Johns Hopkins Press, Baltimore, Md., 1958. x+613 pp. $6.00.

An eight page "Mathematical Summary" and an annotated bibliography of approximately 1000 items on Linear, Nonlinear, and Dynamic Programming. The theory of Constant-Sum Two Person Games is considered as far as it is related to Linear Programming. The principal classification is by subject (Mathematical Theory, Computational Techniques, Methods for Solving Linear Systems, Linear Inequalities and Convex Sets, Game Theory, Applications — by industries). An author (or issuing agency) index refers back to the main entries. The listing aims at completeness up to June 1957.

M. J. Beckmann (New Haven, Conn.).

★ **Goldstein, Allen A.; and Cheney, Ward. Note on a paper by Zuhovickiĭ concerning the Tchebycheff problem for linear equations.** Mathematical Pre-print Series, 8. Convair Astronautics, San Diego, Calif., August, 1957. i+8 pp.

Translation of Dokl. Akad. Nauk SSSR 79 (1951), 561–564 [MR 13, 285].

Brauer, Alfred. A method for the computation of the greatest root of a positive matrix. J. Soc. Indust. Appl. Math. 5 (1957), 250–253.

It is known that the greatest root of a positive matrix is

positive and that it exceeds the diagonal elements. It is further known that the matrix is similar to one with equal row sums, in which case the greatest root is equal to the common values of the row sums. For the computation in the general case, the author starts by replacing the matrix A by $B=A-\mu I$, when μ is the smallest diagonal element. Let R be the greatest row sum of B and r the least, and put $R-r=6d$. It is then shown that a matrix similar to B exists for which the quantity corresponding to $R-r$ does not exceed $4d$. Repeating this process, the characteristic root of B can be obtained to any accuracy. The similarities used are only diagonal matrices of positive elements. *O. Taussky-Todd.*

Kron, Gabriel. Factorized inverse of partitioned matrices. Matrix Tensor Quart. **8** (1957), 39–41.

The author explains a method of solving a set of linear equations by using the well-known inverse of a partitioned matrix. *H. S. A. Potter* (Aberdeen).

Koehler, Fulton. Estimates for the eigenvalues of infinite matrices. Pacific J. Math. **7** (1957), 1391–1404.

The Rayleigh-Ritz method for approximating the eigenvalues of a self-adjoint positive definite operator A consists of determining the eigenvalues of P_nAP_n, where P_n is the orthogonal projection into an n-dimensional subspace of the vector space in which A is defined. If the eigenvalues $\lambda_1\geqq\lambda_2\geqq\cdots$ of A are defined as the successive maxima of the Rayleigh quotient $(Au, u)/(u, u)$, and $\lambda_1^n\geqq\lambda_2^n\geqq\cdots$ are the corresponding eigenvalues of P_nAP_n, it is easily seen that $\lambda_i\geqq\lambda_i^n$, so that the λ_i^n are lower bounds for the λ_i. (The λ_i may be thought of as the reciprocals of the eigenvalues of a differential operator).

Clearly, A may be decomposed into $A=P_nAP_n+(E-P_n)AP_n+P_nA(E-P_n)+(E-P_n)A(E-P_n)$, where E is the identity operator. Therefore, if the norms $\|(E-P_n)AP_n\|$ and $\|(E-P_n)A(E-P_n)\|$ are small, the difference $\lambda_i-\lambda_i^n$ is small. The author derives an explicit upper bound for λ_i in terms of the λ_i^n and the bounds for $\|(E-P_n)AP_n\|$ and $\|(E-P_n)A(E-P_n)\|$. Thus, if these bounds are available, one finds an upper bound for λ_i.

A more accurate upper bound for λ_i is given in terms of λ_i^n, λ_{i+1}^n, $\cdots$, λ_n^n, $\|Au_i^n-\sum_{j=1}^{n}(Au_i^n, u_j^n)u_j^n\|$, $\cdots$, $\|Au_n^n-\sum_1^n(Au_n^n, u_j^n)u_j^n\|$, and $\|(E-P_n)A(E-P_n)\|$, where u_j^n is the normalized eigenvector of P_nAP_n corresponding to λ_j^n. The bound is the largest zero of a polynomial of degree $n-i+2$, and is close to the lower bound when the norms involved are all small.

The results are extended to the more general eigenvalue problem $Au=\lambda Bu$, where both A and B are self-adjoint and positive definite.

Excellent numerical results are obtained in two examples dealing with ordinary differential equations. The application of the results to the two-dimensional membrane problem is sketched. *H. F. Weinberger.*

Rutishauser, Heinz. Zur Bestimmung der Eigenwerte schiefsymmetrischer Matrizen. Z. Angew. Math. Phys. **9b** (1958), 586–590.

If the matrix is not of even order, adjoin a row and column of zeros. Application to this matrix of the rotations of Givens [National Bureau of Standards Applied Mathematics Series, no. 29, U.S. Government Printing Office, Washington, 1953, pp. 117–122; MR **15**, 472] reduces it to a skew-symmetric tridiagonal matrix. The author applies the QD-algorithm [H. Rutishauser, Der Quotienten-Differenzen-Algorithmus, Mitt. Inst. Angew. Math. Zürich, no. 7 (1957), 74 pp.; MR **19**, 686] to a finite continued fraction whose $n/2$ poles are the quantities $-\lambda_i^2$. *A. S. Householder* (Oak Ridge, Tenn.).

Trees, R. E.; and Coleman, C. DeW. Tables for diagonalizing second-order matrices. J. Res. Nat. Bur. Standards **60** (1958), 201–214.

Let (a_{ij}) be a real, symmetric matrix of order two. Let $x=\frac{1}{2}\arctan(2a_{12}(a_{11}-a_{22})^{-1})$. Then the eigenvalues of (a_{ij}) are $a_{11}+a_{12}\tan x$ and $a_{22}-a_{12}\tan x$, and the eigenvectors are $(\cos x \sin x)$ and $(-\sin x \cos x)$. The tables give $\tan x$, $\sin x$ and $\cos x$ to five places, with $a_{12}(a_{11}-a_{22})^{-1}$ and $(a_{11}-a_{22})a_{12}^{-1}$ as arguments with the range 0(.001)1.199. *K. Goldberg* (Washington, D.C.).

Dettmar, H.-K.; und Schlüter, A. Praktische Lösung von Eigenwertaufgaben des Hartree-Fockschen Typs. Z. Angew. Math. Mech. **38** (1958), 220–236.

The paper describes a numerical method for solving the Schroedinger equation pertaining to the many-body problem of quantum mechanics, as applied to the electron shell of the atom, using the Hartree-Fock approximation. The emphasis is on the mathematical treatment and theorems are kept as general as possible.

The Hartree-Fock equation represents an eigenvalue problem involving non-linear, integro-differential equations of the type

$$(H-\lambda)P=LP, \quad P\in\mathfrak{P},$$

where H and L are linear, Hermitian operators and $\mathfrak{P}$ is a linear function space whose elements satisfy certain boundary conditions. An iterative procedure is introduced so that these equations may be replaced by non-homogeneous, linear differential equations

$$(H-\lambda)P=F, \quad P\in\mathfrak{P},$$

where H and $F=LP$ are now fixed. With estimates $\bar{P}$ and $\bar{\lambda}$ of P and λ respectively, these equations are solved in terms of the boundary value problem

$$(H-\bar{\lambda})q_i=q_{i-1}, \quad q_0=F=L\bar{P}, \quad q_i\in\mathfrak{P},$$

and methods are described for finding improved estimates $\bar{P}$ and $\bar{\lambda}$. The entire procedure can be made fully automatic. *C. Froese* (Vancouver, B.C.).

Franklin, J. N. On the numerical solution of characteristic equations in flutter analysis. J. Assoc. Comput. Mach. **5** (1958), 45–51.

Es handelt sich um die explizite Bestimmung des Polynoms $\varphi(\lambda)=\det F(\lambda)$, wo $F(\lambda)$ eine $n\times n$ Matrix ist, mit Polynomen in λ vom Grad $\leqq k$ als deren Elemente. Die hierfür geeignetste Methode von Leverrier erfordert ungefähr $k^4\cdot n^4$ Multiplikationen. Verf. gibt eine Methode an, die nur noch $k^2\cdot n^4$ Multiplikationen verlangt. Es liegt ihr der folgende mathematische Satz zu Grunde: Man bilde aus den $n\times n$ Matrizen $A_1, \cdots, A_k$ rekursiv die weitern Matrizen

$$S_r=S_{r-1}A_1+\cdots+S_1A_{r-1}+rA_r \quad (r=1, \cdots, k),$$

$$S_r=S_{r-1}A_1+\cdots+S_{r-k}A_k \quad (r=k+1, \cdots).$$

Bezeichnet σ_r die Summe der r-ten Potenzen der kn Wurzeln der Gleichung

$$\det(\lambda^kE-\lambda^{k-1}A_1-\cdots-\lambda A_{k-1}-A_k)=0,$$

so ist

$$\sigma_r=S_pS_r \quad (r=1, 2, \cdots).$$

A. Pfluger (Zürich).

Kirkham, Don. Graphs and formulas for zeros of cross product Bessel functions. J. Math. Phys. **36** (1958), 371–377.

This paper gives graphs from which can be read zeros of the equation

$$J_n(x)Y_n(kx)-J_n(kx)Y_n(x)=0$$

and of the similar equations in which either the functions of x, or those of kx, or all functions, are replaced by derivatives. For the first three equations, $n=0(1)3$, and for the last, $n=1(1)4$. All values of $k\geqq 0$ are represented and up to 9 zeros may be obtained in favourable cases, in general $x\leqq 17$, $kx\leqq 29$. The scale of the graphs is small — about 1 decimal is obtainable. Asymptotic formulae of the McMahon type are given, and other methods suggested for improving accuracy, together with references to tables of solutions. *J. C. P. Miller.*

Hart, Roger G. A formula for the approximation of definite integrals of the normal distribution function. Math. Tables Aids Comput. **11** (1957), 265.

Fox, L.; and Mitchell, A. R. Boundary-value techniques for the numerical solution of initial-value problems in ordinary differential equations. Quart. J. Mech. Appl. Math. **10** (1957), 232–243.

Il a été proposé à diverses reprises [D. N. de G. Allen et R. T. Severn, même J. **4** (1951), 209–222; MR **13**, 287; L. Fox, ibid. **7** (1954), 367–378; MR **16**, 1055; etc.] de remplacer un problème de conditions initiales par un problème aux limites d'ordre supérieur. Sans proposer de méthode nouvelle l'auteur discute les procédés autérieurs en se limitant au cas d'une équation linéaire du premier ordre. *J. Kuntzmann* (Grenoble).

Capra, Vincenzo. Valutazione degli errori nella integrazione numerica dei sistemi di equazioni differenziali ordinarie. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 188–203.

The numerical solution of the initial value problem for systems $y_i'=f_i(x, y_1, \cdots, y_n)$ usually consists of a table of values $\bar{y}_{ik}$ representing approximations to the exact values $y_i(x_k)$ of the solutions at certain points $x=x_k$, $x_0<x_1<\cdots<x_m$. The author suggests a general method of estimating the errors $z_{ik}=\bar{y}_{ik}-y_i(x_k)$. He introduces functions $\bar{y}_i(x)$ which connect the values $\bar{y}_{i0}, \bar{y}_{i1}, \cdots, \bar{y}_{im}$ and which have continuous first derivatives inside each subinterval (x_{k-1}, x_k). These functions will show up "directional defects" $|\bar{y}_i'(x)-f_i(x, \bar{y}_1, \cdots, \bar{y}_n)|$. On the assumption that bounds for these defects in the subintervals (x_{k-1}, x_k) are available, as well as estimates for the partial derivatives $\partial f_i/\partial y_j$, the author sets up an auxiliary system of linear differential equations with piecewise constant coefficients such that its solutions $z_i(x)$ have the property $z_i(x_k)\geqq|z_{ik}|$. Several solutions $z_i(x)$ are explicitly written down corresponding to various simplifying assumptions.

The method is illustrated for a single differential equation ($n=1$) in the cases where the numerical integration is done by a method due to Milne and a Runge-Kutta method. The functions $\bar{y}(x)$ here are taken to be certain interpolating functions which arise naturally in connection with these methods. The crucial difficulty of estimating the directional defects is partly overcome by simply calculating the defects at suitable points rather than finding rigorous upper bounds. In this way the author is able to express his error bounds in terms of quantities most of which occur in the integration process. A numerical example shows that the error bound comes out about 10 times larger than the actual error. {Reviewer's note: Error estimations along similar lines are given in W. Uhlmann, Z. Angew. Math. Mech. **37** (1957), 88–99; MR **19**, 180.} *Walter Gautschi.*

Capra, Vincenzo. Nuove formule per l'integrazione numerica delle equazioni differenziali ordinarie del 1° e del 2° ordine. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 301–359.

Numerous formulas of the predictor-corrector type based on osculating interpolation of various orders. There are remarks concerning possible checks, change of step length, convergence of iterations and error estimations, the latter with reference to the paper reviewed above. *Walter Gautschi* (Washington, D.C.).

Uhlmann, W. Differenzenverfahren für die 2. und 3. Randwertaufgabe mit krummlinigen Rändern bei $\Delta u(x, y)=r(x, y, u)$. Z. Angew. Math. Mech. **38** (1958), 236–251.

The author develops formulas for the numerical solution of $u_{xx}+u_{yy}=r(x, y, u)$ in a region G with a curved boundary Γ. In an earlier paper with J. Albrecht [same Z. **37** (1957), 212–224; MR **19**, 884], u was specified along Γ; in the present paper $\partial u/\partial n$ or $\partial u/\partial n+\alpha u$ is specified along Γ. Using Taylor series expansions, the author develops difference approximations when the stars contain more than the usual 5 points; this reduces the truncation error without introducing a great many mesh points. The paper concludes with the solution of Laplace's equation over a semi-circle, with mixed boundary conditions. The greater accuracy obtained when derivatives on boundaries are approximated using more points than the usual two is stressed. *M. A. Hyman.*

Heinrich, H. Bemerkungen zum Verfahren von Ernst Schmidt zur graphischen Integration der Wärmeleitungsgleichung. Z. Angew. Math. Mech. **38** (1958), 70–71.

Èzrohi, I. A. General forms of the remainder terms of linear formulas in multidimensional approximate analysis. II. Mat. Sb. N.S. **43(85)** (1957), 9–28. (Russian)

The author continues his work [Mat. Sb. N.S. **38(80)** (1956), 389–416; MR **18**, 32; see also Sard, [Proc. Symposia Appl. Math., v. 6, McGraw Hill, New York, 1956, pp. 177–185; MR **18**, 33] and obtains integral representations of certain linear functionals in spaces of functions $f(x_1, \cdots, x_n)$ in $a_i\leqq x_i\leqq b_i$, $i=1, \cdots, n$. He uses these to obtain the remainders of interpolation formulas and formulas of mechanic quadratures. Unlike in his previous paper, the spaces considered here are defined in terms of mixed derivatives of f, while derivatives in only one variable were considered before. For example, the space $L_{k_1;s_1,\cdots,s_n}{}^p$ consists of all functions f with derivatives $f^{(i_1,\cdots,i_n)}=\partial^{i_1+\cdots+i_n}f/\partial\xi_1^{i_1}\cdots\partial\xi_n^{i_n}$ existing for $\xi_j=x_j$, $a_j\leqq x_j\leqq b_j$ if $i_j=s_j$, and for $\xi_j=b_j$ if $i_j<s_j$, such that each derivative $f^{(i_1,\cdots,s_k,\cdots,i_n)}(\xi_1, \cdots, x_k, \cdots, \xi_n)$ is integrable

with exponent p in x_k and strongly continuous in all other variables. *G. G. Lorentz* (Syracuse, N.Y.).

Morita, Katuhiko. Nomographical determination of the complex roots of the biquadratic algebraic equations. Mem. Fac. Tech. Kanazawa Univ. **1** (1952), 15–32. (2 inserts)

In this paper, the author continues his work on nomographic solutions of the stability biquadratic equations. In comparison with the previous methods, he states that accuracy of the roots is rather good except in a limited domain of the chart. Here matrix theory of alignment charts is introduced and employed. *E. Frank.*

★**Хованский, Г. С. [Hovanskiĭ, G. S.] Номограммы с ориентированным транспарантом. [Nomograms with an oriented transparency.]** Biblioteka prikladnogo analiza i vyčislitel'noĭ matematiki. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 204 pp. 4.60 rubles.

Vurcfeld, Jaromír. Ein Nomogramm für die Lösung von Gleichungen vierten Grades. Apl. Mat. 3 (1958), 223–232. (Czech. Russian and German summaries)

Džems-Levi, G. E. Nomogram construction without quadratures. Dokl. Akad. Nauk SSSR (N.S.) **115** (1957), 438–440. (Russian)

Džems-Levi, G. E. On nomogramming of equations of 4th nomogrammic order. Mat. Sb. N.S. **44(86)** (1958), 123–130. (Russian)

Gol'cman, F. M. Graphoanalytical method of the frequency analysis of seismic waves. Vestnik Leningrad. Univ. **11** (1956), no. 16, 45–56. (Russian)

The idea of the method is based on approximating a seismic curve by the extrema of trigonometric functions. The following formula is suggested:

$$h(\omega)=2/\pi\cdot\sum(-1)^k A_k t_k e^{-i\omega\theta_k}\cos(\omega t_k/2)/[1-(\omega t_k/\pi)^2],$$

where the summation is taken for $k=0, 1, \cdots, n-1$, n being the number of the extrema of the signal, A_k the amplitude, t_k the halfperiod of the kth extremum, θ_k the distance of the kth extremum from the origin, and ω the cyclical frequency. Graphical and analytical methods of calculations are discussed. *S. Kulik.*

★**Белоусов, С. Л. [Belousov, S. L.] Таблицы нормированных присоединенных полиномов Лежандра. [Tables of normalized associated Legendre polynomials.]** Izdat. Akad. Nauk SSSR, Moscow, 1956. 379 pp. 23.70 rubles.

These are tables of the associated functions $\bar{P}_n^m(\cos\theta)$ defined by

$$\bar{P}_n^m(\cos\theta)=\sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\theta),$$

the normalization resulting from this definition being such that

$$\int_{-1}^{+1}[\bar{P}_n^m(x)]^2dx=1.$$

The functions tabulated are thus identical in every way with those whose behavior is exhibited graphically up to $n=8$ in four diagrams in the second and subsequent editions of Jahnke and Emde [Funktionentafeln mit Formeln und Kurven, 2nd ed., Teubner, Leipzig-Berlin, 1933, pp. 178–179; also with various page numbers in later editions]. Except for the difference of symbolism, they are also those denoted by $F_n^m(\cos\theta)$ in Section 16 of Fletcher, Miller, and Rosenhead [An index of mathematical tables, McGraw-Hill, New York, 1946; MR 8, 286].

The arguments in the Belousov tables are $m=0(1)36$, $n=m(1)56$, $\theta=0(2°.5)90°$. Each pair of values of m, n belongs to one column of 37 entries corresponding to the 37 values of θ. The function values are to 6D, without differences. Nine figures were kept in the computations.

The author refers not only to several well-known tables, but also to tables by Ya. M. Heĭfec [Tables of normalized associated Legendre polynomials, Izdat. Gidrometeorol., Moscow, 1950], which are stated to give values of associated functions for $m=1(1)12$, $n=m(1)20$, and each 5° of θ. On page 15 are given nine corrections (two of them taken from Heĭfec) to numerical values contained in R. and L. Egersdörfer [Formeln und Tabellen der zugeordneten Kugelfunktionen 1. Art von $n=1$ bis $n=20$, T. I, Springer, Berlin, 1936]. A 13 page introduction deals with the applications of the tabulated functions to physical problems, with the method of computation, and with the accuracy.

(Adapted from review of A. Fletcher in Math. Tables Aids Comput. **11**, 276.)

G. E. Forsythe.

★**Dwight, Herbert Bristol. Mathematical tables of elementary and some higher mathematical functions. 2nd ed.** Dover Publications, Inc., New York, 1958. iv+219 pp. $1.75.

This edition is similar to the first edition [MR **2**, 366], which was published by McGraw-Hill Book Co., except that tables in hundredths of degrees have been omitted and a considerable number of tables has been added, including those of a number of types of Bessel functions.

See also: Linear Algebra: Bodewig. **Approximations, Orthogonal Functions:** Luke. **Fluid Mechanics, Acoustics:** Alihaškin. **Quantum Mechanics:** Metropolis, Bivins, Storm, Turkevich, Miller and Friedlander. **Geophysics:** Hristow. **Programming, Resource Allocation, Games:** Bellman and Dreyfus; Bellman; Ford and Fulkerson.

Computing Machines

★**Culbertson, James T. Mathematics and logic for digital devices.** D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1958. x+224 pp. $4.85.

According to the preface, this book was "written to serve as an undergraduate college textbook for mathematics and electronics majors and for other students who plan to enter the computer field. The only prerequisite is college algebra." Chapter headings: I. Some preliminary topics [with emphasis on algorithms]. II. Permutations and combinations. III. Probability. IV. Number systems. V. Traditional logic. VI. Boolean algebra of classes. VII. Boolean algebra of propositions. VIII. Applications to switching circuits. Every chapter contains at least one section on computer applications. There are 1262 exercises interspersed throughout the text.

W. C. Salmon (Providence, R.I.).

Basu, B. K.; and Rao, P. V. S. A modified gating logic to improve the speed of operation of double rank counters. Proc. Indian Acad. Sci. Sect. A. **46** (1957), 354–359.

The concept of preparing carry circuits is applied to the doublerank counter, as a means of reducing the time required for performing a carry. The logic of the circuits is similar to that of the carry-circuits in the early relay computers, in which the slowness of the components made such time-saving logic essential. In the Basu-Rao counter, the carry is propagated over an "external" path through gates, instead of by way of the flip-flops, with considerable saving in time. *G. R. Stibitz.*

Bell, D. A. The principles of sorting. Comput. J. **1** (1958), 71–77.

Methods of sorting suitable for use on electronic computers are described. The accompanying table summarizes their characteristics.

Method	Minimum storage required	Minimum number of steps
Pigeon-hole	$N+K$	N
Radix	$2N$	$N \log K$
Merging	$1.5N$	$N[\log(N+1)/2]$
Repeated comparison	N	$N^2/6$
Insertion	N	$N^2/4$
Address calculation	$2.2N$	N
Selection	N	$N^2/2$
Counting	N	$N^2/2$

Here N=number of items to be sorted, and K=range of key. *C. C. Gotlieb* (Toronto, Ont.).

Glantz, Herbert T. On the recognition of information with a digital computer. J. Assoc. Comput. Mach. **4** (1957), 178–188.

The paper discusses methods of having a computer decide when groups of given characters, called the subject set, are found within an assigned list, called the target set, allowing that errors may be present in the subject set. An example arises when a computer is to determine whether a sequence of letters specifies one of a list of cities. A transformation P, which changes the subject set into the same form as the target set, is defined, and also a recognition operator R, dependent on a comparison parameter and a recognition threshold. Four types of error are defined: (1) A failure to obtain a positive recognition; (2) an ambiguous recognition caused by obtaining more than one positive recognition; (3) an error of the first kind, where a decrease in the level of the threshold parameter R_2 would have allowed a correct recognition; (4) an incorrect positive recognition.

The author believes that a statistical sampling is necessary to determine the main types of errors in the sampling data, and even then he is not hopeful that present techniques are capable of having machines attain a "near human" level of sophistication. *C. C. Gotlieb.*

Gorn, Saul. Standardized programming methods and universal coding. J. Assoc. Comput. Mach. **4** (1957), 254–273.

After analysis of a problem is completed, the steps in programming and coding can be broken down as follows: (a) designing the storage and scaling plan for the main input and output variables; (b) drawing the schematic flow chart for the problem, (c) drawing the detailed flow chart; (d) enumerating the boxes; (e) copying the program from the flow chart, doing the preliminary code, and forming the "address directory" for the program section of the code; (f) listing constants and variables in the program, and forming the rest of the address directory; (g) producing the final code. The terminology and notation which have been developed for programming are clearly defined and techniques for carrying out each of the above steps are described, thereby making possible a universal coding system. In automatic coding, the computer itself carries out some of the above steps; an example is given where a problem prepared by means of the universal code was run on two different computers. *C. C. Gotlieb* (Toronto, Ont.).

Hamblin, C. L. Computer languages. Austral. J. Sci. **20** (1957), 135–139.

Napolitano, Luigi G. Su alcuni problemi concernenti la programmazione di equazioni differenziali ordinarie non lineari. Ricerca, Napoli (2) **8** (1957), Gennaio-Giugno, 60–72.

Kozlov, È. S.; and Nikolaev, N. S. Approximate solution of partial differential equations with the aid of electric models. Avtomat. i Telemeh. **17** (1956), 890–896, appendix to no. 10, 2. (Russian. English summary)

Aleskerov, S. A. Concerning the calculation of electromagnetic systems by means of electrical computers. Avtomat. i Telemeh. **18** (1957), 764–772. (Russian. English summary)

See also: Numerical Methods: Ascher and Forsythe; Kogbetliantz. **Information and Communication Theory:** Brown.

PROBABILITY

Simaika, J. B. On two characteristics of a random direction. Proc. Math. Phys. Soc. Egypt **5** (1956), no. 4, 113–119 (1957). (Arabic summary)

An elaboration of two earlier notes [C. R. Acad. Sci. Paris **241** (1955), 1375–1377; Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 753–756; MR **17**, 500; **18**, 679]. *H. P. McKean, Jr.* (Cambridge, Mass.).

Laha, R. G. An example of a nonnormal distribution where the quotient follows the Cauchy law. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 222–223.

If x and y are independently and identically distributed random variables with density

$$f(t)=\frac{\sqrt{2}}{\pi}\frac{1}{1+t^4} \quad (-\infty<t<\infty),$$

then their ratio has a Cauchy distribution. This shows that the normal distribution is not characterized by the property of Cauchy-distributed ratios. *L. E. Moses.*

Lebedintseva, E. K. On the limiting distributions for normed sums of independent random vectors. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 219–221. (Ukrainian. Russian and English summaries)

It is proved that the class of limiting distributions for

normed sums of independent s-dimensional random vectors — each having as its function of probability distribution one of l given distribution functions — consists of all possible compositions, not greater than l stable distribution functions, if certain additional conditions are satisfied for the examined sequences of random vectors. *Author's summary.*

★ **Mourier, Edith. L-random elements and L^*-random elements in Banach spaces.** Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. 2, pp. 231–242. University of California Press, Berkeley and Los Angeles, 1956. $6.50.

The author sketches her joint work with R. Fortet [Ann. Sci. Ecole Norm. Sup. (3) **70** (1953), 267–285; Bull. Sci. Math. (2) **78** (1954), 14–30; Studia Math. 15 (1955), 62–79; Ann. Inst. H. Poincaré **13** (1953), 161–244; MR **15**, 808; **16**, 149; **16**, 268; and the paper listed below] and indicates how their results can be used to establish R. von Mises' central limit theorem for empirical distribution functions [Ann. Inst. H. Poincaré, **6** (1936), 185–212]. *H. P. McKean, Jr.* (Princeton, N.J.).

Fortet, R.; et Mourier, E. Les fonctions aléatoires comme éléments aléatoires dans les espaces de Banach. Studia Math. **15** (1955), 62–79.

Beck, Anatole. Une loi forte des grands nombres pour des espaces de Banach uniformément convexes. C. R. Acad. Sci. Paris **246** (1958), 696–698.

The following theorem is a generalization of the law of large numbers for random elements of a Banach space as stated by R. Fortet and E. Mourier in the paper listed above.

Theorem: Let X be a uniformly convex Banach space and let $\{X_i\}$ be a sequence of strongly integrable, independent, random vectors of X. Suppose that for all i, $E(X_i)=0$ and $\sigma^2(X_i)=E(\|X_i\|^2)<M$ for some finite M independent of i. Then, almost surely $n^{-1}\sum_{i=1}^{n} X_i$ tends strongly to zero.

The proof is sketched and a counter-example given to delineate the importance of the hypothesis of uniform convexity. *H. P. Kramer* (Murray Hill, N.J.).

Kuipers, L. Some remarks on asymptotic distribution functions. Arch. Math. **8** (1957), 104–108.

Let $x(t)$, where $t=0, 1, \cdots, -1, -2, \cdots$, be a sequence of real numbers, let $N(T, x<\xi)$ be the number of numbers $x(t)$ with $x(t)<\xi$ and $t=-T, \cdots, -1, 0, 1, \cdots, T$, and let $\sigma(T, \xi)=N(T, x<\xi)/(2T+1)$. Then $\sigma(\xi)=\lim_{T\to\infty}\sigma(T, \xi)$ is called the asymptotic distribution function (a.d.f.) of the sequence $x(t)$, provided $\sigma(\xi)<\infty$, $\lim_{\xi\to+\infty}\sigma(\xi)=1$, and $\lim_{\xi\to-\infty}\sigma(\xi)=0$. At discontinuity points of σ, set $\sigma(\xi)=\{\sigma(\xi+0)+\sigma(\xi-0)\}/2$. When $x(t)$ has the a.d.f. $\sigma(\xi)$, the author determines the a.d.f. for the sequence of reciprocals of $x(t)$ and for the sequence of remainders of $x(t)$ modulo 1. These results are the discrete analogs of results for the continuous case obtained by Haviland [Amer. J. Math. **63** (1941), 408–414; MR **2**, 280]. Several other results are given which relate to theorems of Schoenberg [Math. Z. **28** (1928), 171–199]. *H. P. Edmundson.*

Rényi, A.; and Zergényi, E. An inequality for uncorrelated random variables. Czechoslovak Math. J. **6(81)** (1956), 415–419. (Russian summary)

Let $\xi_1, \xi_2, \cdots, \xi_k, \cdots$ be a sequence of uncorrelated random variables with mean values 0 and variances D_k^2. Let c_k denote a non-increasing sequence of positive numbers, satisfying the inequality $1<c\leqq c_k/c_{2k}\leqq C$ $(k=1, 2, \cdots)$. The authors prove that

$$E\left(\sup_{n\leq k} c_k^2 \Big|\sum_{j=1}^{k}\xi_j\Big|^2\right)\leqq K\left(c_n^2\sum_{j=1}^{n} D_j^2+\sum_{j=n+1}^{\infty} D_j^2 c_j^2\log^2 j\right)$$

for $n=1, 2, \cdots$, where the constant K depends only on the constants c and C. This inequality simplifies the proof of the strong law of large numbers for uncorrelated random variables. *J. Wolfowitz* (Ithaca, N.Y.).

Nasr, Saad K. Determination of the Mourier mean of random variables situated in some Banach spaces. Proc. Math. Phys. Soc. Egypt **5** (1956), no. 4, 79–85 (1957). (Arabic summary)

The author determines the Mourier mean $M(x)$ [C. R. Acad. Sci. Paris **229** (1949), 1300–1301; MR **11**, 376] for random variables x with values in the Banach spaces (1) C, of continuous functions on $(0, 1)$; (2) c, of convergent sequences of numbers; (3) $l^{(p)}$; and (4) $L^{(r)}$. Theorem 2, dealing with C, is an example of his results. It is shown that if $x(t)$ is a stochastic process all of whose sample functions are in C such that $E\|x\|<\infty$, then $M(x)=E\{x(t)\}\in C$ and is unique. *H. P. Kramer.*

A. Sales Vallés, Francisco de. On periodic random functions. Collect. Math. **9** (1957), 105–141. (Spanish)

A random function (r.f.) $X(t)$ is called periodic if all of its joint distribution functions are periodic. In Section I, theorem 2 states that a necessary and sufficient condition that $X(t)$ be periodic is that its covariance function $\Gamma(t, t')$ be periodic in both variables. In Section II, the properties of r.f.s derived from a periodic r.f. by integration and moving average are discussed. Section III is concerned with the geometry of $L_2(X)$, the Hilbert space generated by a periodic r.f. $X(t)$. If T_h is the operator of translation through h, $\{T_hZ\}$ is called the trajectory generated by Z. A periodic r.f. has a closed trajectory on $L_2(x)$. It is shown that there exist random variables in $L_2(x)$ with null trajectory. If $X(t)$ is integrable and ω is its period, such an element is the temporal mean

$$\bar{X}=\omega^{-1}\int_0^{\omega} X(t)dt.$$

{The proof given of Theorem 1 is erroneous. The mistake consists in concluding from the periodicity of $X(t)$ that $F(x_0, t; x_1, t+\omega)=F(x_0, t; x_1, t)$. From the definition of periodicity it follows only that $F(x_0, t; x_1, t+\omega)=F(x_0, t-\omega; x_1, t)$.}

H. P. Kramer (Murray Hill, N.J.).

Lévy, Paul. Remarques sur le processus de W. Feller et H. P. MacKean. C. R. Acad Sci. Paris **245** (1957), 1772–1774.

Let $X(t)$ $(t\geqq 0)$ be a one-dimensional Brownian motion, and let $i_{x,\nu}$ $(\nu=1, 2, \cdots)$ be the intervals (on t-axis) whose union constitutes the complementary set to $\{t; X(t)=x\}$. Let $N(t, x, m)$ (and $L(t, x, m)$) denote the number of intervals $i_{x,\nu}$ contained in $(0, t)$ and of length $>m(\leqq m)$. Then it is known [P. Lévy, Compositio Math. **7** (1939), 283–339; MR **1**, 150] that $\lim_{m\to 0} N\sqrt{m}$ and $\lim_{m\to 0} L/\sqrt{m}$ both exist almost certainly and have the same limit $s=s(t, x)$. Let $F(x)$ be a stepwise increasing function such that $F(x)\uparrow\infty$ (as $x\downarrow\infty$) and $F(x)\downarrow-\infty$ (as $x\downarrow-\infty$). By taking $u=u(t)=\int_{-\infty}^{\infty} s(t, x)dF(x)$ as a new time parameter,

we set (1): $X(t)=Y(u)$. It then turns out that the process Y is a process of W. Feller and H. McKean, Jr. [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 351–354; MR **19**, 327], i.e., Y is a homogeneous Markov process whose possible states consist of a denumerable infinity of instantaneous state, realisable on measurable sets of the parameter u. Theorem 2 of this paper concerns the determination, up to an additive constant, of F from Y through (1). According to a letter from the author, the proof of Theorem 2, sketched in this paper, is not correct and he will publish another proof elsewhere. *K. Yosida* (Tokyo).

Koronkevich, A. E. The ergodic theorem for non-stationary random functions. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 231–233. (Ukrainian. Russian and English summaries)

The article deals with the ergodic theorem for the random function $\xi(t)$ belonging to a wider class than that of the random functions which are stationary in the sense of A. Y. Khinchin, viz. for functions whose first moment $M\xi(t)$ is constant, and whose second moment. $M\xi(t+\tau)\xi(t)=B(t;\tau)$, depends on the parameters of t and τ.

Author's summary.

Reich, Edgar. Waiting times when queues are in tandem. Ann. Math. Statist. **28** (1957), 768–773.

Kolmogorov [Math. Ann. **112** (1935), 155–160] gave a necessary and sufficient condition (nsc) for a Markov chain with a finite state-space to be reversible in time. The author extends this result to chains with a denumerably infinite state-space, in the special case where the chain is irreducible positive-recurrent.

The author then obtains a similar nsc for reversibility for continuous-parameter Markov processes which are regular [the system is to be uniquely determined by $q_{jk}=p_{jk}'(0)$], irreducible and of positive-recurrent type. This result is then used to show that all stationary regular 'birth-and-death' processes with a reflecting barrier at zero are reversible in time. An intuitive version of this result was given by P. and T. Ehrenfest [Phys. Z. **8** (1907), 311–314].

The author now turns to unsaturated queues of the type $M/M/s$ [using the notation of the reviewer, Ann. Math. Statist. **24** (1953), 338–354; MR **15**, 44], and shows that the output of such a queue is a Poisson process; he says that this important fact was first discovered by P. J. Burke [Operations Res. **4** (1956), 699–704; MR **18**, 707] by a different method, and mentions also unpublished work by H. H. Goode and R. E. Machol.

Next, tandem queues are discussed. If two $M/M/s$ queues are 'in tandem' (i.e. in series) and if $n_1(t)$ and $n_2(t)$ are the total numbers of persons in each system, then it is shown that $n_1(t_1)$ and $n_2(t_2)$ are independent if $t_2 \leq t_1$, and that if T_1 and T_2 are the total times spent by a customer in each system, then T_1 and T_2 are independent when $s=1$. Some of these results were first given by R. R. P. Jackson [J. Roy. Statist. Soc. Ser. B. **18** (1956), 129–132; MR **18**, 157]. Finally, the author examines the problem of extending his results to $E_j/E_k/s$ queues, but finds that, to a great extent, his results are the best possible. Thus, he shows that the output of an $E_2/E_2/1$ queue is not of type E_2. *D. G. Kendall* (Oxford).

Jackson, James R. Networks of waiting lines. Operations Res. **5** (1957), 518–521.

A network of M multi-server queues is considered, subject to the following assumptions. (1) Queue m contains n_m servers. (2) Customers from outside the network arrive at queue m in a Poisson-type time series with rate λ_m. (3) Once served at queue m (exponential service time with mean $1/\mu_m$), the customer proceeds to queue k, with probability θ_{km}. It is found that, in "equilibrium", the states $\{k_i\}$ (k_i=number of customers waiting and being served at queue i) are distributed independently of each other. As the author points out, this result may be inferred from basic results of Burke and the reviewer [see review above and reference therein]. *E. Reich.*

Stephan, Frederick F. Two queues under preemptive priority with Poisson arrival and service rates. Operations Res. **6** (1958), 399–418.

This paper extends the work of White and Christie [Operations Res. **6** (1958), 79–95; MR **19**, 1091] on preemptive priority queuing. A random walk procedure yields recursive relations between the state probabilities in the steady state. Recursive relations for the moments of the length of the lower priority queue follow. Combined with the use of moment generating functions, this approach provides moments of the waiting time for elements of the lower priority class that arrive when the system is in a given state, i.e. the two queues are of given lengths. The moments of waiting time and time in the system are also obtained recursively for the steady state. Routines for computing and checking are provided. *P. M. Morse.*

Lamens, A.; et Consael, R. Sur le processus non homogène de naissance et de mort. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 597–605.

The authors treat, using generating functions, the birth and death process for a single type of individual, with the birth and death rates and immigration rates time-dependent. Ordinarily, the forward differential equations play the fundamental role [Kendall, Ann. Math. Statist. **19** (1948), 1–15; MR **9**, 451], but the authors base their evaluation of the generating function of the population on the backward equations. *J. L. Doob* (Urbana, Ill.).

Domb, C.; and Fisher, M. E. On random walks with restricted reversals. Proc. Cambridge Philos. Soc. **54** (1958), 48–59.

The authors consider certain restricted random walks over "uniform" lattice points in a finite dimensional space. At each lattice point, a step may be taken in one of q directions. Let $-i$ denote the reverse of direction i and β_i denote the (unconditional) probability of taking a step in the ith direction. The authors restrict their walks by assuming that the conditional probability of a step in the ith direction given that the preceeding step was in the $(-i)$th direction is $\beta_i-\delta$, where $\sum_{i=1}^{q}\beta_i=1+\delta$ and $|\delta|\leq 1$. The double (over n and r) generating function for the probabilities $q_n(r)$ of such a restricted random walk reaching the lattice point r on the nth step is obtained explicitly. This result greatly generalizes the scope of validity of Gillis' result for rectangular lattices [same Proc. **51** (1955), 639–651; MR **17**, 275]. Also the methods used are simpler. By means of this generating function, the authors obtain exact results for the number of closed polygons without reversals and for the mean square lengths of the walks, as well as general asymptotic expansions for the probabilities $q_n(r)$. *R. Pyke.*

★ **Hilibrand, Jack. Characterization of probability distributions for excess physical noises.** Tech. Rep. 276. Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., 1956. 51 pp.

In the paper, theoretical and experimental techniques

are described for characterizing the probability distributions of certain excess physical noises by their moments. In this connection theoretical methods are developed for applying this "moments technique" in the time domain to random-pulse noise, and in the frequency domain to any random functions for which the moments exist. The paper contains a description of an experimental system for measuring the first four moments of noises in the 0.2 cps-10 kc range. It is concluded that under certain circumstances, experimental measurements of moments are more desirable than direct probability density measurements. The theory is illustrated by measurements on $1/f$ noise in germanium diodes.

A. Jensen (Copenhagen).

See also: Partial Order, Lattices: Gulotta. **Measure, Integration:** Dinculeanu. **Topological Groups:** Urbanik. **Computing Machines:** Culbertson. **Statistics:** Steinhaus and Zubrzycki; Grenander. **Structure of Matter:** Hori. **Geophysics:** Pierson. **Economics, Management Science:** Luce; Grenander. **Biology and Sociology:** Leslie; George.

STATISTICS

Herzel, Amato. Sulle poligonali come distribuzioni generatrici di variabili divise in intervalli e sulla correzione dei primi due momenti. Statistica, Bologna **17** (1957), 414–445.

Let $x_i=2id$, for $i=0, 1, 2, \cdots, k$, be equidistant points on the x-axis; $H(n_1, n_2, \cdots, n_k)$ the histogram with the ordinates $n_i \geqq 0$ for $x_{i-1}<x\leqq x_i$, $i=1, 2, \cdots, k$; and $P(y_0, y_1, \cdots, y_k)$ the polygon with the vertices (x_i, y_i), where $y_i \geqq 0$, for $i=0, 1, \cdots, k$. The histogram H is said to be "generated" by the polygon P when $(y_{i-1}+y_i)d=n_i$ for $i=1, 2, \cdots, k$. The author obtains a necessary and sufficient condition which $n_1, \cdots, n_k$ must satisfy in order that there exist a polygon $P(y_0, y_1, \cdots, y_k)$ which generates H. Since, for a given H, there may exist infinitely many generating polygons, the author studies the effect of additional conditions imposed on the polygon, e.g. $y_0=y_k$, or the requirement that P should approximate best in the square mean a given function. He also investigates properties of the Sheppard correction under the assumption that a histogram is generated by a polygon. *Z. W. Birnbaum* (Seattle, Wash.).

Gurland, John. Distribution of the maximum of the arithmetic mean of correlated random variables. Ann. Math. Statist. **26** (1955), 294–300.

Cornish, E. A. The sampling distributions of statistics derived from the multivariate t-distribution. Austral. J. Phys. **8** (1955), 193–199.

Johnson, N. L. A note on the mean deviation of the binomial distribution. Biometrika **44** (1957), 532–533.

Banerjee, Saibal Kumar. A lower bound to the probability of Student's ratio. Sankhyā **18** (1957), 391–394.

The author derives a lower bound for the probability that $\bar{x}+ts/\sqrt{n}\geqq m\geqq\bar{x}-ts/\sqrt{n}$, as a function of t, n and $B_2=3+k_4/k_2^2$. Numerical values are given for $t=3$, $n=4(2)12,20,30$, $B_2=1(1)6,8,10$. *G. E. Noether.*

Siotani, Minoru. The significance of the discordant variance estimates. Ann. Inst. Statist. Math., Tokyo **7** (1955), 39–55.

Basu, D. A note on the theory of unbiased estimation. Ann. Math. Statist. **26** (1955), 345–348.

Barton, D. E. A comparison of two sorts of test for a change of location applicable to truncated data. J. Roy. Statist. Soc. Ser. B. **19** (1957), 119–124.

The object of this paper is to compare the powers of one and two-sample tests of change of center of location based on the difference of sample quantiles with those of the corresponding combinatorial test based on quantile division of the sample. The two sorts of test have the same power in the limit (in both cases), and numerical comparisons in the finite case for normal population are discussed, indicating that the limiting forms give a good approximation to the powers even for quite small sample sizes. *T. Kitagawa* (Fukuoka).

Stuart, Alan. The asymptotic relative efficiencies of tests and the derivatives of their power functions. Skand. Aktuarietidskr. **37** (1954), 163–169 (1955).

Buehler, Robert J. Confidence intervals for the product of two binomial parameters. J. Amer. Statist. Assoc. **52** (1957), 482–493.

For $i=1, 2$ let X_i denote a binomial variable $B(P_i, n_i)$. Assuming the n_i substantial and the P_i small, and using the Poisson approximation to the binomial, the author computes two short tables of "reasonable" sets of upper confidence limits for the product, say $Q=P_1P_2$. This "reasonable" set is defined as follows. For a single binomial variable X, with parameters P and n, and for $k=0, 1, \cdots, n$, denote by $C_n(k, \alpha)$ the lowest upper confidence limit corresponding to the fixed confidence coefficient α. That is, $C_n(k, \alpha)$ is the least upper bound of those values of P for which $P\{X\leqq k\}\geqq\alpha$. Returning to the couple X_1, X_2 of independent binomial variables, let (k_1, k_2) denote their possible sample points. These possible sample points, say S_i, are numbered from $i=1$ to $N=(n_1+1)(n_2+1)$ in increasing order of the product $C_{n_1}(k_1, \alpha)C_{n_2}(k_2, \alpha)$. Now, for any point S_i, the "reasonable" upper confidence limit $C_{n_1,n_2}(S_i, \alpha)$ of Q is defined as the least upper bound of those values of the product P_1P_2 for which the probability of observing either of the i first sample points $S_1, S_2, \cdots, S_i$ is at least equal to α.

J. Neyman (Berkeley, Calif.).

Fieller, E. C. Symposium on interval estimation: Some problems in interval estimation. J. Roy. Statist. Soc. Ser. B. **16** (1954), 175–185.

Crow, Edwin L. Confidence intervals for a proportion. Biometrika **43** (1956), 423–435.

Creasy, Monica A. Symposium on interval estimation: Limits for the ratio of means. J. Roy. Statist. Soc. Ser. B. **16** (1954), 186–194.

Lafon-Augé, Monique. Conditions d'existence d'un bloc incomplet partiellement équilibré. C. R. Acad. Sci. Paris **245** (1957), 1774–1775.

Guttman, Louis. A generalized simplex for factor analysis. Psychometrika **20** (1955), 173–192.

Amato, Vittorio. On the distribution of Gini's G coefficient of rank correlation in rankings containing ties. Metron **18** (1956), no. 1–2, 83–106.

The author has defined a generalized coefficient of rank correlation involving generic measures of dispersion of the two ranked series. By taking particular measures of dispersion he obtains the rank correlation coefficients of Spearman and Kendall and, by taking mean deviations of ranks, a coefficient proposed by Gini in 1914. This paper deals mainly with the definition and distribution of Gini's coefficient when there are ties. While the argument is mainly an informal discussion of particular cases, it is asserted that, even in the case of ties, Gini's coefficient tends to the normal distribution as the length of the series increases, faster than Spearman's coefficient, but slower than Kendall's coefficient. *S. W. Nash.*

Seal, K. C. On a class of decision procedures for ranking means of normal populations. Ann. Math. Statist. **26** (1955), 387–398.

Bechhofer, Robert E.; and Sobel, Milton. A single-sample multiple decision procedure for ranking variances of normal populations. Ann. Math. Statist. **25** (1954), 273–289.

Anscombe, F. J.; and Page, E. S. Sequential tests for binomial and exponential populations. Biometrika **41** (1954), 252–253.

Steinhaus, H.; and Zubrzycki, S. On the comparison of two production processes and the rule of dualism. Colloq. Math. **5** (1957), 103–115.

Silk ribbons are produced by two machines ($i=1, 2$). Let n and m be the number of defects in the first and second ribbon. A classical inspection plan consists of observing the ribbons until $n+m=N$, a prescribed number; a sequential plan consists of observing the first ribbon until n has a prescribed value. Let P stand for probability, W for likelihood. Defects are independently distributed according to Poisson distributions: $P_k(h)=(c_ih)^k \exp(-c_ih)/k!$ is the probability that in each segment $(t, t+h)$, k defects are observed ($i=1, 2$). The fiducial theory is based upon the relation: $W(c<\alpha|\kappa=k)=P(\kappa\geqq k|c=\alpha)$ [R. A. Fisher, Statistical methods and scientific inference, Oliver and Boyd, Edinburgh, 1956]. From the point of view of the classical Bayes approach, this corresponds to a prior distribution of the parameter c which is uniform for $c>0$. This point of view is reminiscent of that held by H. Jeffreys [Theory of probability, 2d ed., Oxford, 1948; for a review of the 1st ed. see MR 1, 151], but does not correspond to the view of Fisher.

The subsequent analysis employs the ratio of the parameters of the Poisson distributions c_2/c_1. The prior distributions assumed for this ratio are as follows: (1) $\alpha/(1+\alpha)$ for positive α; (2) uniform distribution for $\log(1+c_2/c_1)$ (3) uniform distribution for the ratio c_2/c_1. Let p be the probability that the next defect appears on the first ribbon, q that it appears on the second ribbon. $p=c_1/(c_1+c_2)$; $q=1-p$. Hence, our problem is equivalent to independent drawings from an urn containing fractions p and q of red and black balls. The prior probabilities are now: (1) q uniformly distributed in (0, 1); (2) $\log[1/(1-q)]$ uniformly distributed ;(3) $q/(1-q)$ uniformly distributed. Pearson's tables of the incomplete Beta function can be used in order to compute the required probabilities and likelihood. *G. Tintner* (Ames, Iowa).

Moriguti, Sigeiti. Notes on sampling inspection plans. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **3** (1955), 99–121.

Narain, R. D. The general theory of sampling on successive occasions. Bull. Inst. Internat. Statist. **24** (1954), 2ème livraison, 87–89.

Ura, Shoji. Minimax approach to a single sampling inspection by attributes. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **3** (1955), 140–148.

Wise, M. E. Formulae relating to single-sample inspection by attributes. Philips Res. Rep. **10** (1955), 97–112.

Johnson, N. L. Optimal sampling for quota fulfilment. Biometrika **44** (1957), 518–520.

Mandel, John. Fitting a straight line to certain types of cumulative data. J. Amer. Statist. Assoc. **52** (1957), 552–566.

Let $0=x_0<x_1<\cdots<x_n$ be constants, $\varepsilon_1, \cdots, \varepsilon_n$ be independent standardized variables. The paper considers some results of confusing the linear regression models

$$Y_i=\beta x_i+\sigma\varepsilon_i \quad (0<\sigma<\infty),$$

$$Y_i=\beta x_i+C\sum_{\nu=1}^{i}(x_\nu-x_{\nu-1})^k\varepsilon_i \quad (0<C<\infty),$$

mostly for the case $x_i=i$. *J. Hannan.*

★ **Quenouille, M. H. The analysis of multiple time-series.** Griffin's Statistical Monographs & Courses, no. I. Hafner Publishing Company, New York, 1957. 105 pp.

This 100 page monograph is concerned with the specification and analysis of multivariate time series representable by a unilateral moving average of independent random elements. The author discusses the connection between the linear relations governing the process and the covariances, or their formal generating function (the spectral density matrix, although this term is not used). The questions of hypothesis fitting and testing are discussed, as well as those of faulty specification (thoroughly) and of identification (not so thoroughly). The monograph concludes with the numerical analysis of some economic series.

The principal innovation of the book is the use of a certain canonical analysis of the variables (associated with a partial fraction expansion of the operator relating variables and residuals).

The arguments tend to be sketchy (e.g. the use of formal operator series instead of Laurent series with specified regions of validity), but they are buttressed by a wealth of numerical examples, which will be of interest to anybody with an experimental turn of mind.

There are a few misstatements: the estimates r_1/r_2 and $C_sC_{s+1}^{-1}$ mentioned on pages 66 and 71 are not asymptotically efficient, as claimed, nor (cf. p. 79) do the lower

order covariances provide a set of sufficient statistics for the coefficients of a finite moving average. In his rather experimental search (p. 68) for an improved estimate of ρ in a one-variable first-order autoregression, the author seems to have forgotten that the two end observations, together with C_0 and C_1, are jointly sufficient in the Gaussian case considered. Other attempts (cf. pp. 74–5) to derive "improved" estimates seem to require fuller explanation.

However, these quibbles are inevitable, and should not be allowed to obscure the worth of a study which should be of particular value to econometricians who are interested in the actual inter-variable relations, rather than in the frequency characteristics of the series.

P. Whittle (Wellington).

Grenander, Ulf. Modern trends in time series analysis. Sankhyā **18** (1957), 149–158.

An expository article drawing from U. Grenander and M. Rosenblatt [Statistical analysis of stationary time series, Wiley, New York, 1957; MR **18**, 959].

H. Wold (New York, N.Y.).

Shinbrot, Marvin. On the integral equation occurring in optimization theory with nonstationary inputs. J. Math. Phys. **36** (1957), 121–129.

This is a revised treatment, with some modifications, of the problem discussed in NACA Tech. Note no. **3791** (1956) [MR **19**, 71]. The author announces a forthcoming paper in which the restriction that $\sum (a_p(t)b_p(\tau) - a_p(\tau)b_p(t))$ be a function of $t-\tau$ is removed.

N. Levinson (Cambridge, Mass.).

Saunders, K. D. A power-spectrum equation for stationary random gusts, including a sample problem. J. Aero. Sci. **25** (1958), 295–300.

David, S. T. Symposium on interval estimation: Confidence intervals for parameters in Markov autoregressive schemes. J. Roy. Statist. Soc. Ser. B. **16** (1954), 195–203.

See also: Integral Transforms: Wintner. **Probability:** Simaika; Laha. **Economics, Management Science:** Ammeter; Grenander.

PHYSICAL APPLICATIONS

★ **Van Nostrand's scientific encyclopedia. 3rd ed.** D. Van Nostrand Co., Inc., Princeton-Toronto-London-New York, 1958. v+1839 pp. (11 plates) $26.95.

The third edition of this well-known reference book is substantially increased in size over the last.

★ **The international dictionary of physics and electronics.** D. Van Nostrand Co., Inc., Princeton-Toronto-London-New York, 1956. xvii+1004 pp. $22.50.

The terms defined in this book include laws, relationships, equations, basic principles and concepts, as well as the most widely used instruments and apparatus.

The list of contributing editors contains distinguished names.

Mechanics of Particles and Systems

Vodička, Vaclav. Integration of the equations of motion of a particle on an ellipsoidal surface. Arch. Mech. Stos. **10** (1958), 143–146. (Polish and Russian summaries)

The author integrates the equations of motion for motion in a vertical meridian plane of an ellipsoid of revolution, discussed in same Arch. **10** (1958), 107–114 [MR **19**, 1206].

P. Franklin (Cambridge, Mass.).

Parkyn, D. G. The rising of tops with rounded pegs. Physica **24** (1958), 313–330.

The author calculates the forces for a rolling and for a sliding top, and determines which components are principally effective in erecting the top. These are the sliding friction for a sliding top, and the couple at the point of contact opposing rolling for a rolling top. Experimental results are analysed to show that the motion is mainly rolling, but is modified by periodic sliding oscillations.

P. Franklin (Cambridge, Mass.).

Gran Olsson, R. Einfache Darstellung dynamischer und hydrodynamischer Probleme durch Einführung logarithmischer Massstäbe. Z. Angew. Math. Mech. **37** (1957), 149–153.

Vogel, Théodore. Systèmes dynamiques à hérédité non linéaire et à mémoire totale. C. R. Acad. Sci. Paris **246** (1958), 59–61.

Košťál, Rostislav. Les propriétés du mouvement composé de deux oscillations non amorties de la même direction et avec des amplitudes égales. Práce Brn. Českoslov. Akad. Věd **29** (1957), 277–301. (Czech. Russian and French summaries)

Murphy, Charles H., Jr. Criteria for the generalized dynamic stability of a rolling symmetric missile. J. Aero. Sci. **24** (1957), 773–774.

Statistical Thermodynamics and Mechanics

Castañs Camargo, Manuel. Uncertainty and statistical mechanics. Rev. Acad. Ci. Madrid **52** (1958), 63–86. (Spanish)

The author introduces an index of uncertainty γ for each particle, such that the entropy is $S=R\log\gamma$. When applying this equation to ideal gases, γ is chosen in such a way that the result is identical with Boltzmann's $S=k\log P$. However, it is argued that the first formula is preferable, because it shows more clearly why Maxwell's demon is able to lower the entropy owing to its knowledge about the molecules.

N. G. van Kampen (Utrecht).

Pekeris, Chaim L.; and Alterman, Zipora. Solution of the Boltzmann-Hilbert integral equation. II. The coefficients of viscosity and heat conduction. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 998–1007.

In a previous paper [same Proc. **41** (1955), 661–669; MR **17**, 273] it was shown that it is possible to reduce the Boltzmann-Hilbert integral equation, occurring in the classical problem of transport phenomena in a rigid-sphere gas model, into a differential equation. In the case of self-diffusion treated there, this differential equation was of the second order, and its solution yielded a value for the coefficient of self-diffusion which was in good

agreement with the value obtained by the variational Chapman-Enskog method. In this paper the method is applied to the problems of heat conduction and viscosity. In both cases the differential equations for the respective distribution functions are of the fourth order. The solution of these equations leads to values for the coefficients of heat conduction and of viscosity which are in good agreement with the values obtained by the Chapman-Enskog method. From the tabulated values of the distribution functions it follows that the linearized approximation becomes relatively poorer in the outer regions of momentum-space. A differential equation of the fourth order for the distribution function in the case of viscosity was derived by L. Boltzmann. Boltzmann's differential equation is incorrect as it stands because of errors that crept into the last stages of his derivation. Boltzmann did not integrate the differential equation. To the authors' knowledge the differential equation governing the distribution function in the case of heat conduction, which is derived and solved in this paper, is new. (Author's summary.) *J. F. Heyda* (Cincinnati, Ohio).

Kadomtsev, B. B. On the effective field in a plasma. Soviet Physics. JETP **6** (1958), 117–122.

The Bogoliubov equations for the "partial distribution functions" are used to compute the effective field acting on charged particles in a plasma. It is shown that the effective field differs from the mean field by a small quantity of the order $1/N$, where N is the number of particles within a sphere whose radius is equal to the Debye radius. This result also holds in the presence of a magnetic field. *Author's summary.*

Lax, Melvin. Generalized mobility theory. Phys. Rev. (2) **109** (1958), 1921–1926.

An equation is derived for the conductivity tensor by applying perturbation theory to the density matrix equation for the system consisting of electron plus scattering system in an applied field, by assuming the applied field to be weak. If the coupling between the electron and the scattering system is weak, the usual expression for the conductivity tensor is obtained. A relation between a many-electron treatment and the one-electron treatment used is proved for both Fermi-Dirac and Boltzmann statistics, assuming however, that the Hamiltonian is separable so that correlations occur for Fermi-Dirac systems because of statistics.

D. ter Haar (Oxford).

Klimontovich, Iu. L. On the method of "second quantization" in phase space. Soviet Physics JETP **6** (1958), 753–760.

From the general Liouville equation for the many-particle distribution function, transport equations are derived by averaging for the one- and two-particle distribution functions. Instead of the momenta and coordinates, the number of particles in different points of mu-space are used as independent variables. These variables are random functions, and the one and two-particle distribution functions occurring in the transport equations are averages of these variables or of their products. *D. ter Haar* (Oxford).

Kohn, W. Effective mass theory in solids from a many-particle standpoint. Phys. Rev. (2) **105** (1957), 509–516.

Instead of using an independent particle model to derive the effective mass equation for an extra hole or electron in an insulator or semi-conductor, this paper starts from the exact many-body Hamiltonian. Use is made of the translational properties of both the Hamiltonian and its eigenfunctions, and it is assumed that there is an energy gap between the ground state and all other states with zero momentum. A small point charge is then assumed to be placed at the origin and, using ordinary perturbation theory, the usual single-particle hydrogen-like effective mass equation is derived for the modulating function which describes the state of the crystal with the extra hole (or electron) and the impurity centre at the origin. *D. ter Haar* (Oxford).

Kohn, Walter. Interaction of charged particles in a dielectric. Phys. Rev. (2) **110** (1958), 857–864.

This paper is a sequel to the paper reviewed above. Using Feynman diagram techniques and the linked cluster theory of Brueckner [ibid. **100** (1955), 36–45] and Goldstone [Proc. Roy. Soc. London. Ser. A. **239** (1957), 267–279; MR **18**, 975], it is shown that the static dielectric constant, defined in terms of the force between two distant point charges immersed in the medium, provided the charges are small or their distance apart large, is equal to the effective dielectric constant which determines the force between an extra hole (or electron) and a distant point charge. It is suggested that the following general charge renormalisation theorem holds for a dielectric: Any pair of extra charges, provided they are sufficiently far apart, and whether carried by electrons, holes, foreign particles (e.g., protons) or classical charge points, interact with each other via the same Coulomb interaction as in vacuo, except that each charge must be renormalised by a factor equal to the reciprocal square root of the static dielectric constant. *D. ter Haar* (Oxford).

Arinstein, E. A. On the statistical theory of crystallization. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 1189–1191. (Russian)

Arinstein, E. A. The crystallization phenomenon in statistical physics. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 615–618. (Russian)

Mori, Hazime; and Ross, John. Transport equation in quantum gases. Phys. Rev. (2) **109** (1958), 1877–1882.

The usual quantum mechanical treatments of irreversible processes invariably contain a logical jump from probability amplitudes to probabilities. To justify this jump is one of the main problems of quantum statistical mechanics. For instance, in the Uehling-Uhlenbeck theory of gases the two-body scattering amplitude is squared to give the scattering cross-section; possible effects of phase relations are therefore neglected, in the same way in which Boltzmann's theory neglects possible correlations between collisions. The aim of the present paper is to justify this procedure for the case of dilute gases.

For this purpose an ensemble is introduced, described by a Wigner distribution function, and the equation of motion is solved using a number of assumptions. The most crucial assumption is that the two-particle distribution function does not depend on the distance between the two particles until they are within the range of the interacting force. Moreover, Boltzmann's Stosszahlansatz is used to specify the dependence on the two momenta. In this way the transport equation of Uehling and Uhlenbeck is obtained. *N. G. van Kampen* (Utrecht).

Prigogine, I.; and Toda, M. On irreversible processes in quantum mechanics. Molecular Phys. 1 (1958), 48–62.

"Weakly coupled" systems are considered, that is, it is supposed that the Hamiltonian can be separated into an unperturbed part and a perturbation term λV. The usual equation for the density matrix is solved to second order in λV, making use of Van Hove's separation into singular and non-singular parts [Physica **21** (1955), 517–540; MR **17**, 115]. The result is a set of equations, each of which only involves those elements of the density matrix which lie on one parallel to the main diagonal. In particular, for the diagonal elements the equation is the Pauli equation and the derivation is identical with Van Hove's. It is shown that in the classical limit the result coincides with the one obtained by Prigogine and Philippot [ibid. **23** (1957), 569–584; MR **19**, 592]. When applying the theory to an oscillator, subject to a random force, the usual formulas of Langevin and Lorentz are obtained.

N. G. van Kampen (Utrecht).

Falkoff, David. Statistical theory of irreversible processes. I. Integral over fluctuation path formulation. Ann. Physics **4** (1958), 325–346.

The theory of time dependent fluctuations can be developed from a purely phenomenological point of view, once it has been assumed that the fluctuations constitute a stationary Gaussian Markov process. In this paper the author first reviews the various properties of such processes. Next, the formulation of Onsager and Machlup in terms of path integrals is discussed and compared with Feynman's treatment of the Schrödinger equation, and with the theory of the Wiener integral. It is pointed out that the Gaussian character of the process is due to the linearity of the phenomenological equations, and hence corresponds to the harmonic oscillator in the quantum-mechanical analogy. It is suggested that the path integral formulation may lead to an extension of fluctuation theory to the case of non-linear phenomenological equations.

N. G. van Kampen (Utrecht).

Klimontovich, Iu. L.; and Temko, S. V. Quantum kinetic equation for plasma with account of correlation. Soviet Physics. JETP **6** (1958), 102–104.

Ziel der Arbeit ist die quantenmechanische Verteilungsfunktion für ein System von Partikeln, im Falle von Coulombscher Wechselwirkung, anzugeben. Dabei müssen Korrelationseffekte berücksichtigt werden und eben in dem sind die Resultate der Verfasser von den schon bekannten verschieden.

Für zentrale Wechselwirkungen werden zuerst die Differentialgleichungen für $F_1(q, p, t)$ und $F_2(q_1, q_2, p_1, p_2, t)$ angegeben, letztere enthält auch eine Funktion F_3. Wenn man sich auf die Korrelation von einzelnen Paaren beschränken kann, so folgt

$$F_2(q_1, q_2, p_1, p_2, t)=F_1(q_1, p_1, t)F_1(q_2, p_2, t)+g(q_1, q_2, p_1, p_2, t),$$

wo g bei schwachen Wechselwirkungen recht klein ist.

Weiter werden für den Fall, dass die zur Austauschwechselwirkung gehörende Korrelationsdistanz klein im Verhältnis zu der der Coulombschen Wechselwirkung ist, Ausdrücke für $\partial F_1/\partial t$ und $g(q, p_1, p_2, F_1)$ angegeben. g sollte man als Funktion von F_1 ausdrücken können, da jedoch das schwierig ist, so wird g nach einem kleinen Parameter entwickelt.

Im Grenzfall geht das erhaltene Resultat in dem von N. N. Bogoliubov und K. P. Gurov [Ž. Eksper. Teoret. Fiz. **17** (1947), 614–628; MR **11**, 147] über, für $h\to 0$ erhält man daraus die Fokker-Plancksche Gleichung für ein Plasma und im stationären Fall folgt daraus einfach die Fermische bezw. Bosesche Verteilung.

T. Neugebauer (Budapest).

Glansdorff, P.; et Passelecq, J. Sur les transformations irréversibles voisines d'un état stationnaire pour des contraintes à courants constants. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 188–194.

It was shown by P. Glansdorff and I. Prigogine [Physica **20** (1954), 773–780] that in a stationary distribution of currents the local entropy production $\sigma=\sum X_i J_i$ is a minimum. Here the J_i are the fluxes and the X_i the generalized thermodynamic forces. They also obtained the differential relation $\sum \delta X_i \delta J_i \geq 0$, if the variation of the forces vanishes on the surface of the volume considered. The present paper extends the differential relation for the case of other boundary conditions, in particular, for that where the variation of the currents vanishes on the surface.

L. Tisza (Cambridge, Mass.).

Prigogine, I.; et Henin, F. Équation de Liouville et section efficace. Acad. Roy. Belg. Bull. Cl. Sci. (5) **43** (1957), 814–827.

The perturbation method developed by Brout and Prigogine [Physica **22** (1956), 621–636] for systems of weakly interacting particles is applied to an ensemble of two particle systems. The change of the relative velocity distribution in a given time interval is expressed as an operator acting upon the initial velocity distribution. The relation of this operator to the usual collision integral is discussed.

S. Prager (Brussels).

See also: Fluid Mechanics, Acoustics: Kraichnan. **Quantum Mechanics:** Stratonovich; Namiki and Iso; Takahashi and Umezawa. **Astronomy:** Ogorodnikov.

Elasticity, Plasticity

★ **Sneddon, I. N.; and Berry, D. S. The classical theory of elasticity.** Handbuch der Physik, herausgegeben von S. Flügge. Bd. 6. Elastizität und Plastizität, pp. 1–126. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. DM 145.00.

The material on the linear theory of elasticity contained in Volume VI of the old Handbuch der Physik comprises four separate articles occupying some 380 pages. In contrast, the entire subject is now disposed of in a single article of 126 pages. This economy was, one suspects, prompted by a decision to make room elsewhere in this volume for the extensive developments in continuum theories appropriate to inelastic media that have taken place since the appearance of the original Handbuch. Yet progress in classical elasticity theory has not been dormant either. The drastic reduction in the space allotted to this subject has rendered the task of the present authors even more difficult than it would otherwise have been. It has forced them to depart from the traditional scope of a major encyclopedic treatment and settle for "a brief survey of certain parts of the basic theory with sufficient discussion of special problems to give some indication of the mathematical techniques available for the solution of such problems."

The opening section of the article is devoted to the analysis of strain and stress as such and to a discussion of the stress-strain law. The succeeding section deals with the problem of Saint-Venant torsion as well as with the pure flexure of beams. There follows a fairly comprehensive division on the two-dimensional problem, in which the main emphasis is placed on complex-variable methods and Fourier-transform techniques. In the division on three-dimensional elastostatics, one finds elements of the theory of stress functions and a short discussion of Betti's method of integration, as well as a subdivision on the application of integral transforms to the solution of three-dimensional problems. Next, there is a brief section concerned with certain aspects of wave propagation and of boundary-value problems in elastokinetics. The article concludes with three pages on thermoelasticity.

The approach employed in the leading section on basic theory is highly formal, and even mathematically sophisticated readers may wonder whether the use of the general tensor algorism has not been carried beyond the point of diminishing returns. In addition, one might question whether the nonlinear formulations which are taken as a point of departure here are justified in the present context, especially since the authors display little patience with the frequently subtle questions involved in the transition to the linear theory. Also, it seems regrettable that basic theorems, as, for example, the uniqueness theorems and the minimum energy principles, have not found their way into these pages.

With respect to the remaining sections, one notes that elastic instability and the theory of plates and shells have joined the victims of the complete reorganization of this volume. Furthermore, in selecting subject matter from elasticity theory proper, the authors have followed their own interests freely. As a result of this natural inclination, some of the most important topics and applications are not mentioned. This lends the entire article and its supporting references a somewhat whimsical character. Among the most conspicuous omissions in the bibliography at the end are the distinguished chapters on elasticity theory in Volume VI of the earlier Handbuch der Physik which include the famous expository contribution by Trefftz. These chapters are supplemented in some useful ways, but hardly superseded, by the article under consideration. *E. Sternberg.*

★ **Jessop, H. T. Photoelasticity.** Handbuch der Physik, herausgegeben von S. Flügge, Bd. 6. Elastizität und Plastizität, pp. 127–228. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. DM 145.00.

★ **Drucker, D. C. Variational principles in the mathematical theory of plasticity.** Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. VIII, pp. 7–22. McGraw-Hill Book Co., Inc., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1958. 153 pp. $7.50.

This paper reviews proofs of some of the known extremum and variational principles for the infinitesimal deformation of elastic-plastic and rigid-plastic continua characterized by either incremental or total strain laws. The intention is to formulate a synthetic approach by which these and future principles could be derived more or less automatically. No reference is made to a previously published systematic method which is both simpler structurally and more powerful [Hill, J. Mech. Phys. Solids **5** (1956), 66–74; MR **18**, 431]. *R. Hill.*

★ **Hodge, P. G., Jr. Discussion of D. C. Drucker's paper "Variational principles in the mathematical theory of plasticity".** Calculus of variations and its applications. Proceedings of Symposia in Applied Mathematics, Vol. VIII, pp. 23–26. McGraw-Hill Book Co., Inc., New York-Toronto-London, for the American Mathematical Society, Providence, R. I., 1958. 153 pp. $7.50.

This short note touches on the well-known fact that the extremum in certain variational principles for rigid-plastic solids is absolute but not analytic. One of the consequent difficulties in obtaining approximate solutions by this means is illustrated by an example on the bending of a plate. *R. Hill* (Nottingham).

Olszak, Wacław; and Perzyna, Piotr. Extremum theorems in the theory of plasticity of non-homogeneous and anisotropic bodies. Arch. Mech. Stos. **9** (1957), 695–712. (Polish and Russian summaries)

The proofs of well-known extremum principles for work-hardening elastic-plastic and rigid-plastic solids are repeated at length with emphasis on the fact that they are valid when the plastic potential is anisotropic and varies with position in the body. Since such a situation was already taken into account in the existing proofs in a general manner, the present paper appears superfluous. *R. Hill* (Nottingham).

Atsumi, Akira. Stresses in a plate under tension and containing an infinite row of semi-circular notches. Z. Angew. Math. Phys. **8** (1957), 466–477.

This paper deals with the determination of the stresses in a semi-infinite isotropic plate, with a row of an infinite number of semicircular notches in the edge, under the action of a tension at infinity parallel to the notched edge. Numerical results are given. *R. M. Morris.*

Reichel, Alex. Boundary stresses in an infinite hub of special shape. J. Proc. Roy. Soc. New South Wales **91** (1957), 109–119 (1958).

This paper makes use of the notation and results of Muskhelishvili [Some basic problems of the mathematical theory of elasticity, Noordhoff, Groningen, 1953; MR **15**, 370] to find the stresses on the boundary of a hub, considered as an infinite plane with a hole in the shape of an equiangular curvilinear triangle. The stresses are caused by a moment, applied to a rigid kernel fitted into the hole. Muskhelishvili solves the fundamental problem of plane elasticity when the stresses are given on the boundary for the case of a finite region mapped onto a unit circle by a polynomial. The general method used here is a modification of the above method for the case of an infinite region when the displacements are given on the boundary, and when the infinite region is mapped onto unit circle by a function of the form $z=c_0\zeta^{-1}+c_1\zeta+c_2\zeta^2+\cdots$. (Author's summary) *R. M. Morris.*

Seika, M. The stresses in an elliptic ring under concentrated loading. Z. Angew. Math. Mech. **38** (1958), 99–105. (German, French and Russian summaries)

In this paper the problem of an isotropic plate in the form of an elliptic ring bounded by two confocal ellipses, under the action of a pair of compressive forces applied at the ends of the major axes of the outer boundary, is treated by complex variable methods. Numerical results are given. *R. M. Morris* (Cardiff).

Šerman, D. I. On one method of solving torsion, bending and plane elasticity problems for multi-connected regions. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 363–377. (Ukrainian. Russian and English summaries)

This paper presents an effective method of solving some torsion, bending and plane elasticity problems for multi-connected regions. The account of the method accepted in the present paper is simpler and more convenient for the solution of practical problems than the one presented in an earlier paper [Dokl. Akad. Nauk SSSR (N.S.) **55** (1947), 1180–1183; MR **9**, 121]. For simplicity and brevity the argumentation is presented for the case of a double-connected region. To test the efficiency of this method, we entered upon and to a considerable extent advanced the consideration of the problem of the torsion of a prismatic hollow beam with a cross-section in the form of a double-connected region limited externally by a regular curvilinear quadrangle (approaching closely to a square) and internally by a symmetrically located circle. *Author's summary.*

Skripkin, V. A. Approximate formulas for the rotation vector for a small deformation. Prikl. Mat. Meh. **21** (1957), 715. (Russian)

Ivlev, D. D. On the determination of displacements in Galin's problem. Prikl. Mat. Meh. **21** (1957), 716–718. (Russian)

Flitman, L. M. On a boundary problem for an elastic half-space. Izv. Akad. Nauk SSSR. Geofiz. **1958**, 105–106. (Russian)

It is known that the functions φ and ψ appearing in the solution $\mathbf{u}=\operatorname{grad}\varphi+\operatorname{rot}\psi$ of the Lamé elastodynamic equations satisfy, separately, a wave equation, but they enter the boundary conditions in such a manner that separate boundary-value problems cannot be formulated. It is shown that for a plane boundary this may be obtained provided that, for $z=0$, either (1) normal stress, along z-axis, and the displacements, along the x- and y-axes, or (2) two shearing stresses and the displacement, along the z-axis, are prescribed, in terms of x, y and the time. *J. Nowinski* (Madison, Wis.).

Blackburn, W. S. Second-order effects in the torsion and bending of transversely isotropic incompressible elastic beams. Quart. J. Mech. Appl. Math. **11** (1958), 142–158.

The author gives a rather comprehensive treatment of torsion and bending of cylindrical bodies made of homogeneous, incompressible, transversely isotropic elastic material in terms of a second order theory based on nonlinear elasticity theory. Included are general formulae for fractional elongation in torsion and change of length of the line of centroids in bending. The work is well correlated with similar treatments of isotropic materials. *J. L. Ericksen* (Baltimore, Md.).

Alumyaè, N. A. Stability of the equilibrium of a helicoidal shell. Prikl. Mat. Meh. **21** (1957), 823–826. (Russian)

★ **Zyczkowski, Michal. Theory of finite deflections of elastic-plastic beams.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 24–32. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

Euler's elastic problem is generalized by admitting the possibility of plastic yielding, though not of work-hardening. The conventional engineering approximation for partly plastic rectangular bars in pure bending is made. The calculations of the deflected shape is treated in some detail. *R. Hill* (Nottingham).

Vinogradov, A. I.; and Vasil'ev, V. G. Determination of displacements in bar systems of box and I-cross-sections beyond the limit of elasticity. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 409–419. (Ukrainian. Russian and English summaries)

The elastic-plastic flexure of beams and frames is considered in this paper. Under the assumption that the material is perfectly plastic and the bending-moment diagram may be taken a linear function in each interval of division of the beam, a method is proposed and formulas are obtained for finding the elastic-plastic displacement of any point of the systems. *From the author's summary.*

Boley, B. A.; and Tolins, I. S. On the stresses and deflections of rectangular beams. J. Appl. Mech. **23** (1956), 339–342.

Di Taranto, R. A. A method for determining the flexural effects of statically loaded beams on multiple elastic supports. J. Appl. Mech. **23** (1956), 503–508.

Archer, F. E.; and Kitchen, E. M. Stresses in single-span deep beams. Austral. J. Appl. Sci. **7** (1956), 314–326.

Amenzade, Yu. A. Torsion of a prismatic beam with square cross-section, braced by a circular rod. Soobšč. Akad. Nauk Gruzin. SSR **18** (1957), 271–278. (Russian)

Hatiašvili, G. M. Elastic equilibrium of a compound cylindrical bar under a lateral load decreasing along a generator. Soobšč. Akad. Nauk Gruzin. SSR **18** (1957), 393–400. (Russian)

Amenzade, Yu. A. Bending of a prismatic beam weakened by a circular cavity. Akad. Nauk Armyan. SSR. Izvestiya. Fiz.-Mat. Estest. Teh. Nauki **10** (1957), no. 3, 47–63. (1 insert) (Russian. Armenian summary)

★ **Kuhn, Paul. Stresses in aircraft and shell structures.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1956. xx+435 pp. $14.00.

The aim of this book, aside from the formal presentation of stress analysis methods, is the discussion of the physical assumptions by which the analysis of extremely complicated airplane structures can be reduced to relatively simple mathematical problems. The mathematics is on the undergraduate level and the book's mathematical interest may be as a source of large systems of differential and algebraic equations which are generally suitable for machine computation. Some interesting analogies between stiffened sheet structures and electrical networks are also discussed. The book is written primarily for practicing engineers and contains a great deal of information on the analysis of stiffened sheet construction that has not previously been available in book form.

S. Bodner (Providence, R.I.).

Ting, L.; and Yuan, S. W. On radial deflection of a cylinder of finite length with various end conditions. J. Aero. Sci. **25** (1958), 230–234.

Using a simplification of the Flügge equation for the bending of circular cylindrical shells, the author solves the problem of diametrically opposed concentrated radial forces. The basic eighth order linear differential equation differs from the simple Donnell equation used previously by the author and others by eliminating the restriction that only the high circumferential harmonics are important. It would be of general interest to obtain the energy expression corresponding to the basic equation.

The author first obtains the solution for an infinitely long shell which agrees well with that obtained from the Flügge equation. Using this solution as the particular integral, he obtains solutions from the homogeneous equation and is able to satisfy the conditions corresponding to free, simply supported and clamped edges. The results are in good agreement with other more restrictive solutions over their range of applicability. A useful discussion of the appropriateness of various shell equations is also presented. *S. Bodner* (Providence, R.I.).

Reissner, Eric. A note on membrane and bending stresses in spherical shells. J. Soc. Indust. Appl. Math. **4** (1956), 230–240.

Consider a segment of a thin spherical elastic shell acted upon by edge forces which are tangent to the shell surface only. Since no normal component of edge forces and edge stress couples are present, it may appear that the state of stress in the interior of the shell is essentially a membrane state and that, by comparison, the bending stresses are negligible. (In previous work on the problem for the determination of the state of stress by the membrane theory, only part of the boundary conditions is satisfied.) The author shows that a more satisfactory solution is possible and determines the membrane state of stress in the interior of the shell with the aid of one single boundary condition "which involves both the normal stress distribution and the shear stress distribution along the edge of the shell, without explicit reference to bending". Also, it is found that, subject to certain restrictions, deviations from the membrane state "are confined to a narrow edge region within which bending action is of importance", and that "the state of stress outside this edge region is not necessarily a membrane state".

P. M. Naghdi (Berkeley, Calif.).

Mainzer, F.-J. Die Spannungsverteilung in der Haut eines kugelförmigen Behälters bei tangentialer Krafteinleitung durch äquidistante Stützen in einem beliebigen Breitenkreis nach der Membrantheorie. Ing.-Arch. **26** (1958), 81–92.

Determination of stresses in a complete thin spherical shell acted upon by (i) rotationally symmetrical fluid pressures and snow loads and (ii) non-rotationally symmetrical tangential surface loads along either a meridional circle or a meridional strip. The latter load system represents the effect of the external supports of the shell. For the problem as stated, a solution is obtained within the framework of membrane theory of thin shells. It is shown that the non-rotationally symmetric part of the solution contains a sufficient number of constants of integration to allow satisfaction of all membrane stress transition conditions along the support circle or along the support strip. No consideration is given to the corresponding membrane deformations, and accordingly no transition conditions for displacements are considered. {While it is possible that a consideration of the complete problem by means of bending theory will lead to deviations from the membrane solution which are confined to relatively narrow zones adjacent to the edges of the supporting strip, it is also possible that no such confinement exists, and that the actual state deviates appreciably from the membrane state, as determined in the paper, throughout the walls of the shell. For a discussion of related matters see the paper reviewed above.}

E. Reissner (Cambridge, Mass.).

Reissner, Eric. Symmetric bending of shallow shells of revolution. J. Math. Mech. **7** (1958), 121–140.

The author develops a general theory for finite displacements of elastic, nonisotropic shallow shells of revolution (shells having the shape of a shallow bowl) under any rotationally symmetric distributed forces and moments. Temperature strains are included. The shell is assumed to be thin, and transverse shear and normal strains are neglected. The author succeeds in reducing the theory to two simultaneous second order non-linear differential equations in the change in slope in the radial direction and a stress variable representing the radius times the radial internal force per unit length of section.

As one application of this general theory, the author considers edge conditions in isotropic shallow spherical shells under internal pressure. As another, he considers small displacements of orthotropic shallow spherical shells under uniform loading, and finds that the usual membrane approximation for such a case may fail in the case of orthotropy. *L. H. Donnell* (Chicago, Ill.).

Conway, H. D. The flexure of infinite rectangular plates of varying thickness. Ing.-Arch. **26** (1958), 143–145.

A solution is given for a cantilever plate of infinite length in the fixed edge direction, with flexural rigidity varying exponentially in the other direction and subjected to harmonically varying load on the free edge. The author's mention of gear teeth as an application is misleading, since the approximations of classical plate theory are retained, which is hardly adequate for the length-thickness proportions of gear teeth.

L. H. Donnell (Chicago, Ill.).

Langhaar, Henry L.; and Boresi, Arthur P. Snap-through and post-buckling behavior of cylindrical shells under the action of external pressure. Univ. of Illinois Bull. Eng. Exper. Station Bull. Ser. no. **443** (1957), 40 pp.

This appears to be the final report on the authors' significant work on finite-displacement shell theory based on energy considerations, and its application to buckling of thin cylinders under hydrostatic pressure. This is a difficult problem which no one has studied without many approximations; the present work is no exception, but the approximations differ greatly from those made elsewhere and the work is in some ways more exact and in others more approximate than previous studies. The authors derive and use relatively exact expressions for membrane and bending strains; they use three terms in the series expansion in the circumferential direction of not only the important radial displacement and the less important "circumferential" displacement (actually the authors use a tangential rather than a circumferential displacement — the two must be distinguished in finite displacement theory), but also the still less important axial displace-

ment. They consider boundary conditions on all three displacements.

On the other hand, they make the arbitrary assumption that circumferential membrane strains are zero, use only one term in the longitudinal expansion of the radial displacement, neglect prebuckling deformations, and choose the number of circumferential waves to minimize the buckling pressure rather than the potential energy (a procedure which is readily justified in small-displacement buckling, but which needs justifying here). It is reassuring that the final results are consistent with those obtained using very different approximations.

The authors neglect defects, but seem to consider that they have taken some account of them by following Tsien's "snap through" hypothesis. Serious objection can be made to this on the ground that the very considerable reduction in strength of actual specimens below the classical stability value certainly depends on the magnitude (not merely the existence), of the defects and this magnitude varies greatly, especially with the thickness-radius ratio. And we can hardly expect to know anything about these effects without actually studying them, inelegant though such studies may have to be.

L. H. Donnell (Chicago, Ill.).

Gran Olsson, R. The rectangular plate, supported on two edges, with linearly varying thickness in the direction of these edges. Norske Vid. Selsk. Forh., Trondheim **30** (1957), 77–78. (Norwegian)

The author derives a fourth order differential equation for the case mentioned in the title and gives a series solution for the displacement of the plate.

D. J. Hofsommer (Amsterdam).

Sonntag, G. Lange Zylinderschale mit nicht achsensymmetrischer Randbelastung durch Kräfte in Zylinder-Längsrichtung. Z. Angew. Math. Mech. **36** (1956), 289–291.

Postnov, V. A. Large deflections of a plate whose edges are fixed unsymmetrically. Trudy Leningrad. Korablestroitel. Inst. **16** (1955), 21–33. (Russian)

Bassali, W. A.; and Dawoud, R. H. Thin circular plates under certain distributions of normal loading. Mathematika **3** (1956), 144–152.

Olesiak, Zbigniew. A bent circular plate with linear supports inside the plate region. Arch. Mech. Stos. **9** (1957), 227–245. (Polish and Russian summaries)

Scherer, A. Einflussflächen einer Dreiecksplatte mit Aufpunkt am freien Rand. Ing.-Arch. **25** (1957), 255–272.

Voitsekhovskaya, K. F. On the stability of a rectangular plate compressed beyond the limits of elasticity. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 121–125. (Ukrainian. Russian and English summaries)

The stability of a rectangular plate compressed beyond the elasticity limits is considered in the light of the theory of simple loading and the theory of complex loading. The relationships of the stresses and strains in the plate due to the loss of stability are given in the elastic and plastic region. The boundary of the elastic and plastic region is found. Equations are given for a plate subjected to bending as well as expressions for the critical flexibility of the plate. *Author's summary.*

Kazantseva, G. E. On oscillations in thin round plates. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 242–246. (Ukrainian. Russian and English summaries)

This paper deals with the solution of certain boundary problems of the dynamics of round plates of constant and variable thickness. The application of operational calculus and the theorem of reciprocity of work permits writing down certain integral equations and investigating the forced oscillations induced by a perturbing force varying in time by an arbitrary law. *Author's summary.*

Mechovrišvili, Š. S. On infinitesimal bending deflections of toroidal shells. Soobšč. Akad. Nauk Gruzin. SSR **18** (1957), 521–527. (Russian)

Klyušnikov, V. D. A method for solution of the bending problem for elastically clamped plates of arbitrary shape. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 9, 126–130. (Russian)

Galasi, A. A. On stresses in a plate supported by a thin elastic rod. Užgorod. Gos. Univ. Naučn. Zap. **18** (1957), 109–119. (Russian)

Šerman, D. I. On elastic equilibrium of a plate supported at the edge. Akad. Nauk Armyan. SSR. Izvestiya. Fiz.-Mat. Estest. Teh. Nauki **10** (1957), no. 3, 35–46. (Russian. Armenian summary)

Livšic', Ya. D. Calculation of flexible plates with transverse load and forces in the plane of the plate. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 387–399. (Ukrainian. Russian and English summaries)

The author considers a flexible rectangular plate clamped on two edges and freely supported on two others. There are distributed transverse loads and forces in the plane of the plate, distributed along the freely supported edges. The solution is based on the general procedure of calculating flexible plates by the method of nets. *From the author's summary.*

Grigorenko, Ya. M. Recurrent ratios for logarithmic solutions in the bending problem of circular plates. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 400–408. (Ukrainian. Russian and English summaries)

The author considers linear differential equations with variable coefficients in the form of polynomials, to which certain problems of the bending of circular plates can be reduced. These equations are solved in series form in the neighbourhood of a regular singular point.

Recurrent ratios are presented in the paper for the logarithmic solutions of linear differential equations when the roots of the defining equation are equal or differ by integers.

An example is given of the solution of a homogeneous equation in the bending problem of a circular plate of linearly variable thickness with k nodal diameters under the influence of an arbitrary load. *Author's summary.*

Slezinger, I. N. On the bending problem of a flexible rectangular plate. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 460–466. (Ukrainian. Russian and English summaries)

Boĭm, A. A. On stresses in an infinite compressed plate weakened by a trapezoidal or vaulted aperture, with an edge reinforced by an elastic ring. Akad. Nauk Ukrain. RSR. Prikl. Meh. 3 (1957), 471–476. (Ukrainian. Russian summary)

Martinovič, T. L. Bending of a plate bounded by two confocal ellipses with reinforced edges. Akad. Nauk Ukrain. RSR. Prikl. Meh. 4 (1958), 70–79. (Ukrainian. Russian and English summaries.

Kärrholm, Gunnar. Influence functions of elastic plates divided in strips. Chalmers Tekn. Högsk. Handl. no. 201 (1958), 18 pp.

The partial differential equation of elastic plates is solved approximately by dividing the plate in strips and substituting finite differences for derivatives with respect to one of the variables. Influence functions applicable to various boundary conditions are derived to facilitate the calculations. Their use is demonstrated upon a square plate with three edges simply supported and one edge clamped. *Author's summary.*

Grilic'kiĭ, D. V. Mixed boundary problem of elasticity theory for an orthotropic block with a circular notch. Akad. Nauk Ukrain. RSR. Prikl. Meh. 3 (1957), 378–386. (Ukrainian. Russian and English summaries)

This paper deals with the mixed plane problem of the theory of elasticity for an orthotropic body with a circular notch, when the displacement components are given for part of the contour L_1, and the stress components are given for the rest of the contour L_2; the latter are taken equal to zero, without affecting generality. The problem is reduced to a solution of two singular integral equations with variable coefficients.

From the author's summary.

Berry, J. G.; and Reissner, E. The effect of an internal compressible fluid column on the breathing vibrations of a thin pressurized cylindrical shell. J. Aero. Sci. 25 (1958), 288–294.

The authors calculate frequencies of free oscillation of thin cylinders containing a compressible fluid, using the concept of an apparent mass of fluid added to the mass of the cylinder wall, calculated to account for the presence of the fluid. As this concept suggests, the simple cylinder frequencies are lowered by the presence of the fluid; higher frequencies not occurring in simple cylinders are also found, corresponding to the presence of nodal circles in the fluid.

The cylinder theory used is derived from the senior author's theory of finite displacements of shallow shells, by first linearizing it and then treating the portions of the complete cylinder between nodes as shallow shells; because of this, vibrations involving less than four nodes around the circumference cannot be studied. The advantage, if any, of this rather roundabout derivation over the use of well known simplified linear cylinder theory having similar limitations is not discussed.

L. H. Donnell (Chicago, Ill.).

Kantham, C. Lakshmi. Bending and vibration of elastically restrained circular plates. J. Franklin Inst. 265 (1958), 483–491.

Classical plate theory is applied to circular plates having elastic edge restraint defined by $\alpha=\lambda aM/D$, where λ is a constant and α, a, M, D are the edge slope, radius, edge moment and bending rigidity. Closed solutions are found for bending under constant lateral pressure and for free lateral vibration. The author presents interesting curves showing the effect of λ on the deflection under load, and on the frequencies of the first four vibration modes. *L. H. Donnell.*

Cox, Hugh L. Vibration of axially loaded beams carrying distributed masses. J. Acoust. Soc. Amer. 30 (1958), 568–571.

The author considers the transverse vibrations of a straight elastic beam, carrying a distributed mass located symmetrically about the center of the beam, and loaded by axial forces on the ends. The beam is assumed to rest on an elastic foundation, and its ends are either pinned or clamped. The added mass is assumed to have no influence on the stiffness of the beam, and thus none on its potential energy. By using an approximate displacement function, and equating the resulting expressions for maximum kinetic and potential energy, the author obtains formulas for the natural frequencies ω_n in terms of the various parameters of the problem. It is found that $1/\omega_n$ is a linear function of the ratio of added mass to beam mass. A similar problem, in which the added mass does stiffen the beam, thus increasing the potential energy as well as the kinetic energy, has been discussed by H. Cohen and G. H. Handelman [same J. 27 (1955), 177]. *W. E. Boyce* (Troy, N.Y.).

Movčan, A. A. On vibrations of a plate moving in a gas. Prikl. Mat. Meh. 20 (1956), 211–222. (Russian)

Tiffen, R. Dilational and distortional vibrations of semi-infinite solids and plates. Mathematika 3 (1956), 153–163.

Kobrinskiĭ, A. E. On the damping of vibration by impact. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1957, no. 5, 15–29. (Russian)

Conway, H. D. An analogy between the flexural vibrations of a cone and a disc of linearly varying thickness. Z. Angew. Math. Mech. 37 (1957), 406–407.

Federhofer, Karl. Erzwungene Schwingungen eines Kreisringes. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa 166 (1957), 1–13.

★ **Richardson, A. S., Jr. Bending-torsion flutter sensitivity in incompressible and supersonic flow.** Proceedings of the Third Midwestern Conference on Solid Mechanics, 1957, pp. 206–220. University of Michigan Press, Ann Arbor, Mich., 1957. vi+250 pp. $5.50.

Stepanov, R. D. On the flutter of cylindrical shells and panels moving in a gas flow. Prikl. Mat. Meh. 21 (1957) 644–657. (Russian)

Bondar', N. G.; Timošenko, V. V.; and Vysočin, B. M. Free oscillations of triple-articulation parabolic arches. Akad. Nauk Ukrain. RSR. Prikl. Meh. 3 (1957), 467–471. (Ukrainian. Russian summary)

Alumyae, M. O. Linear problems of the theory of static stability and flexural vibrations of thin elastic shells. (Review). Akad. Nauk Ukrain. RSR. Prikl. Meh. 4 (1958), 3–18. (Ukrainian)

Review paper with 156 references.

Strahov, G. I. Dynamic stability of a rod under the action of a load with polar orientation. Latvijas PSR Zinātnu Akad. Vēstis **1957**, no. 3(116), 153–166. (Russian. Latvian summary)

The author assumes that the dynamic stability of elastic systems depends on the behaviour of the load in the process of deformation, as is the case in statical stability. The author solves the problem of dynamic stability of a rod when the vector of the resultant of external forces is polarized (oriented towards some fixed point). The corresponding problem when the resultant of loads is oriented permanently as the tangent to the elastic line, i.e., when it follows the angular displacement, was considered by V. V. Bolotin (1953). The present author finds that the rod can be either in stable equilibrium or in resonance, depending on the position of the above mentioned fixed point, and concludes that with the adequate orientation of external loads, as necessary, the parametric vibrations can be eliminated, or their amplitudes reduced. *T. P. Anđelić* (Belgrade).

Rabotnov, G. N.; and Shesterikov, S. A. Creep stability of columns and plates. J. Mech. Phys. Solids **6** (1957), 27–34.

This paper discusses stability based on the strain hardening hypothesis for creep of Rabotnov [IXe Congrès Internat. Méc. Appl., Actes, t. VIII, Univ. de Bruxelles, 1957, pp. 288–292]. In contrast to previous investigations of creep stability [N. J. Hoff, J. Roy. Aero. Soc. **55** (1954), 3–52; Aero. Quart. **7** (1956), 1–20, and V. I. Rozenblyum, Inz. Sb. **18** (1954), 99–104], the present paper treats the stability problem in a direct sense, i.e., no initial deviation from straightness in a strut or plane form in a plate is assumed. Explicit results for buckling of a strut and of a plate in uniform compression are obtained for a simple special form of creep law. *W. T. Koiter* (Delft).

Shuleshko, P. Buckling of rectangular plates uniformly compressed in two perpendicular directions with one free edge and opposite edge elastically restrained. J. Appl. Mech. **23** (1956), 359–363.

Shibaoka, Yoshio. On the buckling of an elliptic plate with clamped edge. I. J. Phys. Soc. Japan **11** (1956), 1088–1091.

Postnov, V. A. Stability and work beyond the limits of stability of thin plates reinforced by longitudinal ribs. Trudy Leningrad. Korablestroitel. Inst. **15** (1955), 26–41. (Russian)

Fel'dman, M. R. Stability of rods of variable cross-section. Akad. Nauk Armyan. SSR. Izvestiya. Fiz.-Mat. Estest. Teh. Nauki **10** (1957), no. 4, 19–28. (Russian. Armenian summary)

Gazis, D. C.; and Mindlin, R. D. Influence of width on velocities of long waves in plates. J. Appl. Mech. **24** (1957), 541–546.

The paper considers the velocities of propagation of long flexural and extensional waves in a plate or bar in the transition region between the states of generalized plane stress and plane strain; i.e., for arbitrary width-thickness ratios of rectangular sections. The mathematical treatment uses the first-order equations, with the plate displacement expressed in terms of three potentials, derived by T. R. Kane and R. D. Mindlin [J. Appl. Mech. **23** (1956), 277–283]. *G. B. Warburton* (Edinburgh).

Thomas, T. Y. The decay of waves in elastic solids. J. Math. Mech. **6** (1957), 759–768.

This paper investigates the decay in the strength of the irrotational and equivoluminal waves during propagation, on the basis of the author's recent work on extended compatibility conditions [same J. **6** (1957), 311–322; MR **19**, 192]. The author derives the differential equation determining the variation of strength of each type of wave and, by integrating these equations, shows that the strength of an irrotational or equivoluminal wave, propagated in an isotropic medium, will approach zero monotonically as the time of propagation increases indefinitely if, and only if, the wave surface is (1) a developable surface (but not a plane) or (2) a surface of positive curvature and (3) the direction of propagation, which is given by the unit normal to the surface, is such that the mean curvature is negative. *I. N. Sneddon.*

★ **Abramson, H. N.; Plass, H. J.; and Ripperger, E. A. Stress wave propagation in rods and beams.** Advances in applied mechanics, Vol. V, pp. 111–194. Academic Press Inc., New York, N.Y., 1958. x+459 pp. $12.00.

This paper reviews the theoretical treatments of elastic waves in bars, and compares the results predicted by various approximate theories with those derived from the Pochhammer-Chree treatment. It also discusses experimental results on elastic waves. A similar review of plastic wave propagation in extension and flexure is given for thin rods and beams, and the effect of strain-rate is discussed. Elastic-plastic and rigid plastic analyses of flexural problems are outlined and the results compared. The authors do not attempt to discuss viscoelastic wave propagation, but otherwise give a thorough review of the field and cite 145 references to original papers. *H. Kolsky* (London).

Folk, Robert; Fox, George; Shook, C. A.; and Curtis, C. W. Elastic strain produced by sudden application of pressure to one end of a cylindrical bar. I. Theory. J. Acoust. Soc. Amer. **30** (1958), 552–558.

The problem considered is that of determining the elastic strain in a semi-infinite cylindrical bar ($z \geqq 0$, $r < a$) on the assumption that, on the lateral surface $r = a$, $\tau_{rr} = \tau_{rz} = 0$, while at the plane end $z = 0$, $\tau_{zz} = -P_0 S(t)$, $u_r = 0$, where P_0 is a constant and $S(t)$ is a step-function in time. The method of solution consists in using Fourier sine and cosine transforms to remove the z-terms from the equations of motion of the bar, and a Fourier exponential transform to remove the time variable t. In this way the differential equations of motion are reduced to a single ordinary differential equation which can be solved easily in terms of Bessel functions. The use of the appropriate Fourier inversion theorems results in exact expressions for the strain components, in the form of Fourier integrals whose integrands have the form of Pochhammer-Chree waves. These integrals do not lend themselves to direct evaluation, but asymptotic forms which describe the state of strain in the bar at large distances from its plane end are obtained by means of the method of steepest descents. *I. N. Sneddon* (Glasgow).

Gladwell, G. M. L. On the solution of problems of dynamic plane elasticity. Mathematika **4** (1957), 166–168.

It is shown that dynamic plane elasticity problems, where the disturbance travels with constant velocity along a straight line, can be treated in the same way as

for corresponding problems in static aelotropic plane strain. In this way full advantage can be taken of the usual complex variable analysis. *E. H. Mansfield.*

Sneddon, Ian N. Note on a paper by J. R. M. Radok. Quart. Appl. Math. **16** (1958), 197.

It is shown that the solution given in Radok's paper [same Quart. **14** (1956), 289–298; MR **18**, 349) follows from one given earlier by Sneddon [Rend. Circ. Mat. Palermo (2) **1** (1952), 57–62; MR **17**, 802].

Davids, Norman; and Kumar, Sudhir. Cylindrical stress waves in flat slabs. Quart. J. Mech. Appl. Math. **10** (1957), 465–481.

Skuridin, G. A. On jumps of discontinuous solutions of dynamic equations of the theory of elasticity. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 625–633. (Russian)

Petrašen', G. I.; and Uspenskiĭ, I. N. On propagation of waves in an elastic medium consisting of isotropic levels. I. Leningrad. Gos. Univ. Uč. Zap. **208**, Ser. Mat. Nauk **30** (1956), 58–141. (Russian)

Nakada, Osamu. Theory of viscoelasticity of amorphous polymers. II. A note on the time-temperature reducibility of the relaxation and retardation spectra. J. Phys. Soc. Japan **12** (1957), 1218–1225.

The establishment of "master" curves for viscoelastic properties by means of the time-temperature superposition principle is discussed in terms of relaxation and retardation spectra. The molecular theoretical basis for this procedure, which has been established as an empirical method of analysis, is examined. As an application, different types of relaxation spectra for polymers are discussed. *B. Gross* (Rio De Janeiro).

Postolnik, Y. S. Dynamics of a heavy loaded viscoelastic thread (rope) of variable length on a deep level lift. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 186–195. (Ukrainian. Russian and English summaries)

With reference to two earlier papers [G. N. Savin, Ukrain. Mat. Ž. **6** (1954), 126–139; MR **16**, 1060; Yu. S. Postol'nik, Dopovidi Akad. Nauk Ukrain. RSR **1955**, 341–343], the well-known system of equations which describe approximately the motion of the mechanical system mentioned in the title of the paper is given in Section 1. In Section 2, an approximate analytic expression is derived for strains which arise in an viscoelastic thread (rope) during the entire cycle of lifting a load according to a trapezoidal tachogram. This expression is obtained by asymptotic integration of the above mentioned system [see S. F. Feščenko, Nauk. Zap. Kiïv. Derz. Ped. Inst. **6**, Fiz. Mat. Ser. **3** (1948), 406]. In Section 3, the behaviour of the strains is investigated in terms of the viscosity and the velocity of lifting the load. Criteria are given for the damping of the dynamic as well as the complete strains. In Section 4, the results obtained in Section 3 are illustrated graphically. *E. Leimanis.*

Adkins, J. E. Dynamic properties of resilient materials: constitutive equations. Philos. Trans. Roy. Soc. London. Ser. A. **250** (1958), 519–541.

The author developes a general, properly invariant, nonlinear theory of ideal materials for which the stress is a function of displacement and velocity gradients. He allows for aeolotropy as well as holonomic or anholonomic constraints on the motion. Explicit forms of the stress deformation relations for orthotropic and transversely isotropic materials are worked out. *J. L. Ericksen.*

Onat, Emin Turan; Schumann, Walter; and Shield, Richard Thorpe. Design of circular plates for minimum weight. Z. Angew. Math. Phys. **8** (1957), 485–499.

A rigid-plastic circular plate (sandwich or solid) is simply supported at its edge or built in and is loaded transversely by a rotationally symmetric distribution of pressure. The problem is to determine the most economical thickness profile such that deflection is just about to occur under the given loading. The Tresca yield condition and flow rule are adopted, and the calculations are based on the bounds for minimum weight design given by D. C. Drucker and R. T. Shield, Quart. Appl. Math. **15** (1957), 269–281 [MR **19**, 790] *R. Hill* (Nottingham).

Madejski, Jan. Work hardening, elastic after-effect and residual stresses from the point of view of the dynamical theory of plasticity. Rozprawy Inż. **5** (1957), 455–478. (Polish. Russian and English summaries)

Using equations of the author's unpublished dynamical theory of plasticity (which makes the reading of the paper rather difficult) a polycrystalline material is analyzed as an aggregate of grains (a) and intercrystalline "hard" substance (b). In the isothermal case, the heat generated by the occurence of slips in (a) results in transformation of (a) into (b), the increase of (b) being assumed as an extension of the layers of (b). A quasi-uniaxial state of stress is considered and a nonlinear integro-differential equation for the difference of mean stress in (a) and (b) is found. Examples concerning linear and harmonic variation and sudden removal of load are discussed. The model considered reveals cold working- and elastic after-effects as well as residual stress phenomena.

J. Nowinski (Madison, Wis.).

Kumar, Sudhir; and Davids, Norman. Elastic-plastic analysis of scabbing in materials. J. Franklin Inst. **265** (1958), 371–383.

The authors review one-dimensional elastic-plastic wave transmission and the use of associated distance-time and dimensionless stress-strain charts in predicting peak stresses, and discuss application of this to study of fractures under shock loading. Their emphasis on the need for data on material properties under high straining rates is certainly justified since they neglect necking effects, which would greatly complicate analysis if present and would indicate radically changed material behavior if absent. *L. H. Donnell* (Chicago, Ill.).

Johnson, W.; Mellor, P. B.; and Woo, D. M. Extrusion through single hole staggered and unequal multi-hole dies. J. Mech. Phys. Solids **6** (1958), 203–222.

Results of the senior author [L. C. Dodeja and W. Johnson, same J. **5** (1957), 267–280; MR **19**, 907] have been extended to the cases indicated in the title. An approximate graphical method for obtaining the pressure required for extrusion is presented. The example of combined direct and side extrusion illustrates how the solution of a seemingly complex problem may be obtained from the elements of simpler problems.

P. G. Hodge, Jr. (Chicago, Ill.).

Eason, G. Velocity fields for circular plates with the von Mises yield condition. J. Mech. Phys. Solids **6** (1958), 231–235.

The perfectly plastic circular plate problem for any given yield condition is easily reduced to one equilibrium equation, one yield condition, and one flow law equation for the determination of two bending moments and the radial velocity. The first two equations contain only bending moments. The flow law contains only the velocity for a linear yield condition; otherwise it contains both velocity and moments.

A previous paper by Hopkins and Wang [same J. **3** (1955), 117–129; MR **16**, 649] gave the moment solution for the von Mises yield condition. The present paper regains this solution by a different method and then obtains the velocity field from the flow law. The results are compared with those for the Tresca condition; they are found to be qualitatively similar except at the plate center. Here the Tresca condition predicts a discontinuous slope as opposed to a continuous one for the von Mises condition.

The reviewer notes that the single flow law equation is obtained by elimination of a positive parameter μ from two equations. Once the problem has been solved for moments and velocity, μ can be determined. The proposed solution is acceptable only if μ so determined is, in fact, non-negative throughout the plate. The author fails to mention this point, leaving the validity of his solution still open to possible question.

P. G. Hodge, Jr.

Pavlyuk, N. F. The dynamic expansion of a plastic cylinder to partial destruction. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 443–450. (Ukrainian. Russian and English summaries)

Nowacki, W. State of stress in infinite and semi-infinite elastic spaces due to the action of an instantaneous heat source. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 121–130. (Ukrainian. Russian and English summaries)

The problems considered are those of determining the state of stress in an infinite, and then a semi-infinite, elastic solid when there is a source of heat situated within the solid. The author takes the classical differential equation for the potential of thermoelastic deformation (i.e., the Goodier equation) and finds the particular solution corresponding to the temperature distribution due to a point source of heat of constant strength. This implies that the processes are regarded as being quasi-static, the effects of the inertial terms in the equations of motion and of Biot's coupling term in the equation of heat conduction being neglected — as they may be in most problems of physical interest. From this thermoelastic potential the components of stress are calculated.

By considering two point sources situated (in an infinite space) antisymmetrically in relation to the plane $z=0$, the author derives a stress field which is such that the normal component of stress vanishes across the plane $z=0$. By applying Galerkin's displacement function, he then derives a stress field which, when added to this one, ensures that the components of shearing stress on the boundary $z=0$ will also vanish. Finally, the state of stress due to the action of linear and plane instantaneous heat sources in an infinite and in a semi-infinite elastic medium is discussed.

I. N. Sneddon (Glasgow).

Jung, H. Zur Theorie der Wärmespannungen. Wiss. Z. Hochsch. Schwermaschinenbau Magdeburg **1** (1956/57), 15–24.

A steady state temperature field in an elastic isotropic homogeneous body is considered, assuming variation of the modulus in shear G with temperature, and the coefficient of thermal expansion α with temperature and mean normal stress. The last dependence, corroborated by tests, is taken as being linear. Neglecting small terms yields generalized thermo-elastic equations which are solved by an iterative process. A numerical example concerning a thick-walled tube, under axially symmetrical temperature distribution, is given for a linear variation of G with temperature. For temperature dependent α, a similar example has been solved by Hilton (1952), and by the present writer (1953) for α constant and G varying exponentially. *J. Nowinski* (Madison, Wis.).

Madejski, Jan. Theory of similarity of thermo-elasto-plastic phenomena. Rozprawy Inż. **5** (1957), 479–492. (Polish. Russian and English summaries)

On the basis of the author's unpublished dynamical theory of plasticity (dtp), similarity numbers for thermo-elasto-plastic states of bodies are presented. Dimensional representations of several solutions derived in (dtp) are checked, and an example concerning the longtime strength of a superheater, based on experiments of Katz, is given. *J. Nowinski* (Madison, Wis.).

Structure of Matter

★**Seitz, Frederick; and Turnbull, David, editors. Solid state physics. Advances in research and applications. Vol. 4.** Academic Press Inc., New York, 1957. xiv+540 pp. $12.00.

Volume 4 of this "Advances" series begins with a book-length review of "Ferroelectrics and antiferroelectrics" by W. Känzig. Since the theory of ferroelectric and antiferroelectric properties is little understood, in comparison with ferromagnetism, this review is largely a resumé of experimental observations. Thus, dielectric, piezoelectric, elastic and thermal observations on various ferroelectric and antiferroelectric substances are described. The thermodynamical theories of these effects are discussed briefly. A description of optical effects leads to domain structures. Results of diffraction and spectral studies that bear on ferroelectric and antiferroelectric behaviour are given. Finally, there is a review of the molecular models that have been proposed, with emphasis on the difficulties in the way of satisfactory theories.

The second review, almost equally long, on "Theory of mobility of electrons in solids", by F. J. Blatt deals, in contrast to the previous review, with a subject in which theoretical understanding is quite well developed. The one-electron approximation [see J. R. Reitz, in "Solid state physics" vol. 1, Academic Press, New York, 1955; MR **19**, 793; pp. 1–95] and the "effective mass" concept form the basis of the theory, and the effects of electric and magnetic fields and of thermal gradients are considered in detail. The earlier sections deal with the statistics of free electrons and the solution of the Boltzmann transport equation under simple assumptions and also, to a less extent, for more complex situations. The later sections deal in detail with the scattering mechanisms whereby equilibrium of the electron distribution function is achieved, both for metals and semiconductors. There is

some discussion of effects of lack of thermal equilibrium.

A paper on "The orthogonalized plane-wave method", by T. O. Woodruff, is a more detailed treatment of one of the one-electron methods for calculating electron wave functions and energy eigenvalues which were discussed by J. R. Reitz (loc. cit.). The author gives a detailed description of the method and reviews various applications of it that have been made. Finally, he describes his own application of the method in calculating the energy bands of silicon, with special consideration of the sources of error.

R. S. Knox contributes a "Bibliography of atomic wave functions", listed according to element, with an indication of the nature of each computation.

Finally, W. G. Pfann reviews "Techniques of zone melting and crystal growing". He discusses the theory and practice of the removal or uniform distribution ("levelling") of an impurity in a solid by utilizing its different solubilities in the molten and solid phases when a molten zone is moved through the material. The dislocation content of single crystals grown by modern techniques is also discussed briefly. *M. S. Paterson* (Canberra).

Slater, John C. Interaction of waves in crystals. Rev. Mod. Phys. **30** (1958), 197–222.

A review, with a bibliography occupying nearly five pages. The section headings are: 1. General nature of waves in crystals and their interactions; 2. Scattering of an electronic wave by a sinusoidal perturbation; 3. Scattering of an electromagnetic wave by a sinusoidal perturbation; 4. Energy bands in crystals; 5. Thermal vibrations of a crystal; 6. Scattering of X-rays by thermal oscillations; 7. Scattering of electrons and neutrons by thermal oscillations; 8. Compton effect.

A. J. C. Wilson (Cardiff).

Hori, Jun-ichi. On the vibration of disordered linear lattice. II. Progr. Theoret. Phys. **18** (1957), 367–374.

[For part I see Hori and Asahi, same journal **17** (1957), 523–542; MR **18**, 962.] The physical problem can be transformed into the problem of determining the limiting behavior of the elements of matrices obtained as the product of a large number of random matrices. The moments of these elements can be obtained by perturbation techniques in the case of "small" impurities, as in the present paper, or by using Kronecker products, as in Kerner, Proc. Phys. Soc. Sect. A. **69** (1956), 234–244, or in Bellman, Duke Math. J. **21** (1954), 491–500 [MR **15**, 969].

R. Bellman (Santa Monica, Calif.).

Fluid Mechanics, Acoustics

Lighthill, M. J. The fundamental solution for small steady three-dimensional disturbances to a two-dimensional parallel shear flow. J. Fluid Mech. **3** (1957), 113–144.

After a brief review of methods of calculating the flow fields produced by disturbances in rotational basic flows, the author points out a fundamental difficulty in the "secondary-flow" theory (in which the disturbance field is treated as a perturbation of the disturbance field that would occur if the basic flow were uniform): slow attenuation of the secondary-flow disturbance with distance from the obstacle. The author conjectured [same J. **1** (1956), 31–53; **2** (1957), 311–312; MR **18**, 437, 965] that the trouble was caused by nonuniform validity of the approximation sequence in the region far from the obstacle. The analogy with "Stokes' and Whitehead's paradoxes" is mentioned, and a solution analogous to Oseen's is suggested, one in which disturbances, but not the shear, are assumed to be small. In this paper, such a solution is found, and is shown to overlap with the small-shear, secondary-flow solution. The basic flow is a parallel, steady, inviscid, two-dimensional shear flow. The "fundamental solution" due to a weak source is sought.

The method of Fourier transforms is used. Simple solutions are found for a uniformly sheared basic flow (where the result coincides with the secondary-flow solution) and for an exponential basic-flow profile. In the general case it is assumed that the parallel basic flow becomes uniform at $y=\pm\infty$, where the x-axis lies in the flow direction. The character of the solution is determined by studying its Hankel transform, especially for the class of flows where the total variation of the basic stream speed $V(y)$ is small. An interpretation in terms of images, due to M. B. Glauert, is given, and finally the relationship of the present work to theories of the displacement of the stagnation streamline (displacement effect of Pitot tubes) is discussed. *W. R. Sears* (Ithaca, N.Y.).

Bloomer, N. T. Note on the position of ring singularities in an axisymmetric potential field. J. Fluid Mech. **3** (1957), 217–220.

The author's summary reads as follows: In this work a method is given for finding the position of ring singularities in a three-dimensional potential field having axial symmetry, by consideration of the very much easier case of a similar two-dimensional potential function. It is seen that the traces of these ring singularities on a plane through the axis of symmetry occur at points corresponding to those of the singularities existing in the two-dimensional plane when the axial velocity potential functions are the same. It is thought this might be of value in the plotting of stream surfaces.

W. Littman.

Belocerkovskiĭ, S. M. Annular vortex in unsteady motion. Prikl. Mat. Meh. **20** (1956), 173–183. (Russian)

Smith, Austin G. On the generation of the streamwise component of vorticity for flows in rotating passages. Aero. Quart. **8** (1957), 369–383.

Methods have lately been developed [W. R. Hawthorne, Proc. Roy. Soc. London. Ser. A. **206** (1951), 374–387; MR **13**, 177; H. B. Squire and K. G. Winter, J. Aero. Sci. **18** (1951), 271–277] for the approximate calculation of secondary motions (i.e., motions in the plane of the cross-section) in curved channels. The fundamental equation expresses the rate of change in the component of vorticity in the direction of the stream-line in terms of the velocity V, and the total pressure $p+\frac{1}{2}\rho V^2$ which is constant along a stream-line. This work is extended in this paper to the case of rotating channels; the total pressure is no longer constant along a stream-line, but a fundamental equation of the same type can be formed. The increase in the component of vorticity in a simple rotor is calculated approximately, as an example of the method. *W. R. Dean* (London).

Edmunds, D. E. The moving aerofoil in shear flow in the neighbourhood of a plane boundary. Quart. J. Mech. Appl. Math. **10** (1957), 448–464.

The work described in this paper is a natural extension of a previous paper [same J. **9** (1956), 400–424; MR **18**, 530]. The two-dimensional shear flow considered has the form $U=-u_0(1+y_0k/l)$, $V=0$, where u_0 and k are constants and l is a typical length in the aerofoil, e.g., the chord. The x_0-axis lies in the plane wall. Expressions are derived for the forces and the couple acting on a general aerofoil moving in such a shear flow. The results of the previous paper can be obtained as a limiting case by putting $u_0=0$. Results for an aerofoil in unbounded shear flow are also simply obtained as a limiting case. The forces and couple on a flat plate aerofoil are worked out to show the application of the theory. *G. N. Lance.*

Isay, Wolfgang-Hermann. Berechnungsergebnisse der radialen Schaufelgitterströmung. Z. Angew. Math. Mech. **38** (1958), 209–220.

This paper gives the computational details for several examples of radial cascade flow which are studied in accordance with the theory developed in the author's earlier paper [Ing.-Arch. **22** (1954), 203–210; MR **16**, 875]. *M. Marden* (Milwaukee, Wis.).

Alihaškin, Ya. I. A method of calculation of the discharge for flow under pressure into an incomplete crevice. Vyčisl. Mat. **1** (1957), 131–135. (Russian)

The author presents a summation procedure for slowly converging infinite series of the type

$$\sum_{n=1}^{\infty} \frac{K_0(nx)-[I_0(nx)K_0(ny)]/I_0(ny)}{K_1(nx)+[I_1(nx)K_0(ny)]/I_0(ny)} \cdot \frac{\sin^2(n\pi b)/h}{n^3},$$

where $x=\pi r/h$, $y=\pi R/h$, and $I_n(z)$, $K_n(z)$ represent the standard Bessel functions of imaginary arguments. The constants r, R and h are related to the physical configuration of the problem. These types of infinite series are summed by means of the classical transformation methods of Euler and Abel. The results pertaining to a specific physical problem are presented in two tables and a figure. It should be pointed out, however, that there exist more effective methods in modern literature [cf. Shanks, J. Math. Phys. **34** (1955), 1–42; MR **16**, 961].

K. Bhagwandin (Oslo).

Alihaškin, Ya. I. Solution of the problem of an incomplete crevice by the method of lines. Vyčisl. Mat. **1** (1957), 136–152. (Russian)

The author presents an approximate solution of the problem of the flow of liquids into an incomplete crevice. The physical problem is reduced to the mathematical solution of the partial differential equation

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0,$$

subject to the boundary-conditions

$$\left.\frac{\partial\phi(r,z)}{\partial z}\right|_{z=0,\ z=h}=0, \quad \phi(r,z)\Big|_{r=r_0}=0 \quad (0\leqq z\leqq b),$$

$$\left.\frac{\partial\phi(r,z)}{\partial r}\right|_{r=r_0}=0 \quad (b<z\leqq h), \text{ and } \phi(r,z)\Big|_{r=R}=1.$$

The constants h, r_0 and R are related to the physical configuration of the crevice. This problem is solved by means of a certain type of finite difference procedure, called, in the Soviet Union, the method of lines [cf. V. N. Faddeeva, Trudy Mat. Inst. Steklov. **28** (1949), 73–103; MR **12**, 362]. The partial differential equation is reduced to the solution of a system of Bessel-type differential equations. The vanishing of a certain infinite determinant ensures the unique determination of the separation constants. The author presents his results in seven tables and ten figures. From these short tables one can obtain a general set of values for the complete field in the crevice.

K. Bhagwandin (Oslo).

Woods, L. C. On the deflexion of jets by aerofoils. Quart. J. Mech. Appl. Math. **11** (1958), 24–38.

Havelock [Proc. Roy. Soc. London Ser. A **166** (1938), 178–196] solved the problem of an airfoil in a free jet by applying the boundary conditions at the undisturbed free surfaces. The author removes this approximation by mapping the airfoil as a flat plate between parallel streamlines and then applying classical potential theory. His result for the lift-curve slope at zero incidence disagrees with Havelock's. He shows that Havelock's result may be obtained from the present analysis by measuring incidence from the average of upstream and downstream directions, rather than upstream direction, and concludes that this error was introduced by Havelock as a consequence of the misplacement of the free surface boundary condition. *J. W. Miles* (Los Angeles, Calif.).

Iwasa, Yoshiaki. Boundary layer growth of open channel flows on a smooth bed and its contribution to practical application to channel design. Mem. Fac. Engrg. Kyoto Univ. **19** (1957), 229–254.

Roseau, Maurice. Sur le calcul des ondes courbes paralèles au bord dans un liquide pesant sur un fond incliné. C. R. Acad. Sci. Paris **245** (1957), 2472–2474.

Roseau, Maurice. Sur le calcul des ondes courbes paralèles au bord dans un liquide pesant sur un fond incliné. C. R. Acad. Sci. Paris **246** (1958), 53–55.

Homenko, V. S. Motion of a vessel in a shallow channel. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1957**, no. 10, 99–102. (Russian)

Kestin, Joseph; and Newell, Gordon Frank. Theory of oscillation type viscometers: the oscillating cup. I. Z. Angew. Math. Phys. **8** (1957), 433–449.

This is the first in a series of papers dealing with the theory of oscillation type viscometers. In this paper the general problem is formulated. The case of an oscillating cup viscometer (a right circular cylinder of finite height, either partially or completely filled with fluid) is treated in detail. An exact solution (neglecting the non-linear terms in the equations of motion) including transients is derived. The solution is expressed in five different forms suitable for use in different cases, depending upon the geometry of the cup and the properties of the fluid. The significant parameters in the problem are discussed; and conditions which must be satisfied to insure accurate results in practice are given. *R. C. DiPrima.*

Beckwith, David Alan; and Newell, Gordon Frank. Theory of oscillation type viscometers: the oscillating cup. II. Z. Angew. Math. Phys. **8** (1957), 450–465.

This paper, which is a continuation of the previous paper, discusses possible methods of computing the viscosity and density of the fluid using the measured

values of the logarithmic decrement and the circular frequency. Appropriate asymptotic expansions and series which might be used in computations are given for the case of a small cup (the fluid is moving with the cylinder almost as a rigid body), a large cup (only a smaller layer of fluid near the surface of the cup is taking part in the motion), and a cup of intermediate size. No details or results of the actual final computations are included in this paper. *R. C. DiPrima* (Troy, N.Y.).

Feinsilber, A. M. Similarity integrals in the hydrodynamics of heterogeneous and homogeneous processes. Dokl. Akad. Nauk SSSR (N.S.) **112** (1957), 607–610. (Russian)

Consider a slowly moving gas described by the equations of steady plane viscous flow. Suppose that near an obstacle there occurs a heterogeneous chemical reaction whose effects are characterized by (*) $u_x\partial c/\partial x+u_y\partial c/\partial y=\mu(\partial^2 c/\partial x^2+\partial^2 c/\partial y^2)$, where $c(x, y)$ is concentration and μ coefficient of viscosity. Then (*) has the integral $c=a\omega+b$, where ω is vorticity and a and b are constants. In the boundary layer of an obstacle this corresponds to surface activity $(\partial c/\partial y)_0=(a/\mu)dp/dx$. By means of boundary layer considerations the author also finds special integrals of flows with homogeneous reactions characterized by $u_x\partial c/\partial x+u_y\partial c/\partial y=\mu\partial^2 c/\partial y^2+a_0+a_1c+a_2c^2$, where a_0, a_1, and a_2 are constants. *J. H. Giese.*

Struminsky, V. V. Equations of a three-dimensional boundary layer in a compressible fluid for an arbitrary surface. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 271–274. (Russian)

Let S be the surface of an obstacle in a viscous compressible fluid. Let ξ, η, ζ be triply orthogonal curvilinear coordinates such that $\xi=$const, $\eta=$const are normals to S on which ζ is arc length; $\zeta=0$ on S; and $\xi=$const or $\eta=$const are the two families of lines of curvature on S. The familiar distortion $\xi=\xi_1$, $\eta=\eta_1$, $\zeta\sqrt{Re}=\zeta_1$, $u=u_1$, $v=v_1$, $w\sqrt{Re}=w_1$ and passage to the limit for $Re\to\infty$ yield the three dimensional boundary layer equations. The author describes a further transformation of independent and dependent variables, which leads to equations in which the inertial terms of the equations of motion and continuity have the same form as in the three dimensional incompressible boundary layer equations.

J. H. Giese (Aberdeen, Md.).

Niuman, Frank; and Pohlhausen, Karl. Remarks on the paper by M. Finston: 'Free convection past a vertical plate'. Z. Angew. Math. Phys. **9** (1958), 67–69.

In the paper by Finston [same Z. **7** (1956), 527–529; MR **18**, 693], ordinary differential equations for the velocity and temperature distributions were obtained in the case in which the ratio of surface to free-stream temperature is of the form $T_s/T_\infty=1+Bx^\omega$. For a given Prandtl number, these equations involve the two parameters B and ω. The authors of the present paper obtain a one-parameter problem by transformations of the dependent variables and the independent variable, and present numerical and graphical results for values of ω from 0 to 2 and Prandtl number 0.733. *D. W. Dunn.*

Foote, Joe R. An asymptotic method for free convection past a vertical plate. Z. Angew. Math. Phys. **9** (1958), 64–67.

The author obtains solutions of Finston's equations (see previous review) by expanding each of the unknown functions in a series of powers of B. The coefficient functions in these series satisfy linear differential equations with constant coefficients. Analytical solutions are given for the first three pairs of functions.

D. W. Dunn (Ottowa, Ont.).

★ **Narasimhan, M. N. L. On the steady laminar flow of a viscous liquid through an elastic tube with constant temperature gradient.** Proceedings of the Second Congress on Theoretical and Applied Mechanics, New Delhi, October, 1956, pp. 153–164. Indian Society of Theoretical and Applied Mechanics, Indian Institute of Technology, Kharagpur.

Levine, Harold. Skin friction on a strip of finite width moving parallel to its length. J. Fluid Mech. **3** (1957), 145–158.

Regirer, S. A. Some thermohydrodynamic problems concerning a steady one-dimensional flow of a viscous capillary fluid. Prikl. Mat. Meh. **21** (1957), 424–430. (Russian)

Stewartson, K. On asymptotic expansions in the theory of boundary layers. J. Math. Phys. **36** (1957), 173–191.

In each of the four problems considered, the asymptotic form of the stream function $\psi(x, y)$ or of the x-velocity $u(x, y)$ for large values of x (=direction of free flow) is investigated.

In problem (1), the wake behind a flat plate, Goldstein (1933) found that an iteration scheme based on Tollmien's solution (1931) would lead to an inconsistency. In the present paper, the iteration is again based on the b.l. equation written in the form $\partial u/\partial x-\partial^2 u/\partial y^2=F(x, y)$, but its explicit solution in terms of F and the given initial velocity profile $u(0, y)$ [$x=0$ is trailing edge] is now used to discuss Goldstein's set S_n of second order ordinary linear differential equations. This reveals that $u(0, y)$ influences the expansion, not only via Tollmien's term, but also via later terms, and in a much more complicated way. The typical difficulty is that, for certain (infinitely many) values of n, the complementary solution satisfies, "of its own", the two boundary conditions at the b.l. edge, causing the appearance of repeated logarithmic terms. And there remains the difficult problem of finding the corresponding constant factors, presumably determined by $u(0, y)$.

A similar behavior is found in (2), the flat-plate b.l. in hypersonic flow, for which Lees and Probstein (1952) determined the pressure distribution for large x [$x=0$ is the leading edge] in the form $p=\sum a_n x^{-n/2}$; here, the term with $n=3$ depends on the conditions near $x=0$. The present paper raises the question whether it is at all possible to construct the expansion of ψ for large x from the given asymptotic expansion of p. As in (1), for a certain set of values of $n\geqq 2$ (otherwise unknown), the constant factors cannot be determined from the p-expansion alone. For the original problem (2), this means that the term $x^{-3/2}$ should have the factor $a_3+a_3'\log x$, where only a_3' is independent of the neighborhood of $x=0$.

Problem (3) is the b.l. on the outer surface of a semi-infinite circular-cylindrical shell in axial flow, studied by Lighthill and Glauert (1955). Again, the attempt to remove an inconsistency leads to an indeterminacy of a numerical factor, presumably depending on the leading edge neighborhood, which in this case encloses a singular

point, in the standard formulation of b.l. theory. Problem (4) is the flat plate b.l. with constant suction [Iglish (1944) and others]. If the asymptotic expansion of ψ is carried beyond the result of Griffiths and Meredith (1936), a term $Cx^{-3/2}f(y)\exp(-x/4)$ appears, and C cannot be determined from the way in which the problem was set up. However, $C \approx 1.2$ follows by comparison with Iglisch's overall results.

{The reviewer takes the liberty of adding another example: If one tries to expand, in the r, t-plane, the velocity and density of a spherical or cylindrical rarefaction wave advancing towards the center through gas at rest, about the point $(0, t_{refl.})$, where the wave is reflected at the center, one encounters exactly the same difficulty, which in this case, too, prohibits one from predicting a priori the form of the expansion required [Case Institute of Technology, Rep. 3, ONR-contract 1141].}

G. Kuerti (Cleveland, Ohio).

Sparrow, E. M. The thermal boundary layer on a non-isothermal surface with non-uniform free stream velocity. J. Fluid Mech. **4** (1958), 321–329.

For an incompressible fluid with constant viscosity and heat conductivity, the velocity distribution (u, v) in a boundary layer is independent of the temperature distribution, and the latter satisfies $uT_x+vT_y=\alpha T_{yy}$, if frictional heating is neglected. Let u and v be known in the form of Görtler's expansion [J. Math. Mech. **6** (1957), 1–66; MR **18**, 843]. Then in close analogy to Görtler's procedure, $\theta=(T-T_\infty)/(T_w-T_\infty)$ may be sought in the form $\sum \vartheta_n(\eta)\xi^n$, where ξ and η are Görtler's independent variables, T_∞ the constant free stream temperature, and T_w the prescribed wall temperature, which enters the calculation through expansion of $\nu(dT_w/dx)/U(T_w-T_\infty)$ in the form $\sum \lambda^n\xi^n$; and again the functions $\vartheta_n(\eta)$ can be represented as aggregates of "universal" functions (i.e., independent of the input data λ_n). *G. Kuerti.*

Sedney, R. Some aspects of three-dimensional boundary layer flows. Quart. Appl. Math. **15** (1957), 113–122.

By generalizing a method used by C. C. Lin [see Chap. 18 of A. D. Michal, Matrix and tensor calculus, Wiley, New York, 1947; MR **8**, 533], the author writes the boundary-layer equations for incompressible flow, using normal coordinates [as defined by Synge and Schild, Tensor calculus, Univ. of Toronto Press, 1949; MR **11**, 400]. The equations have the invariance properties of "subtensor equations," in the language of Synge and Schild. They reduce to "Cartesian form" in a subclass of surfaces and coordinate systems which is more general than found by Howarth [Phil. Mag. (7) **42** (1951), 239–243; MR **12**, 871]. The conditions for no secondary flow, i.e., for the boundary-layer streamlines to coincide with external-flow streamlines evaluated at the body surface, are determined. For such cases a Mangler-type transformation reduces the equations to the form of plane-flow equations. The displacement surface defined by F. K. Moore [NACA Tech. Note no. **2722** (1952)] is calculated; the result agrees with Moore's. *W. R. Sears* (Ithaca, N.Y.).

Illingworth, C. R. The effects of a sound wave on the compressible boundary layer on a flat plate. J. Fluid Mech. **3** (1958), 471–493.

The principal aim of this paper is to investigate the effect of high wall temperature on the skin friction and heat transfer on a flat plate fixed in a low-speed stream carrying sound waves. The boundary-layer equations are transformed by the method used by F. K. Moore [NACA Tech. Note no. **2471** (1951); MR **13**, 401], which is an extension of the Howarth transformation to unsteady flow. Viscosity is assumed to vary linearly with temperature across the boundary layer at any station. The dissipation term in the energy equation is neglected.

For low-frequency sound waves, solution is carried out in the form of power series in the frequency. The functions that appear in calculating the first-power terms in the expression for skin friction and haet transfer are tabulated for a Prandtl number of 0.72. For high-frequency sound, the expansion is made in inverse powers of the frequency. Again skin friction and haet transfer are worked out, neglecting terms propertional to the $-3/2$ and subsequent powers of the frequency. A comparison is then undertaken between the low- and high-frequency results. There is evidence that the results, together, do not cover the whole frequency range adequately for all ratios of wall to stream temperature. Lighthill, in a related investigation, [Proc. Roy. Soc. London. Ser. A. **224** (1954), 1–23; MR **15**, 907] reached the contrary conclusion for his case, where the ratio was limited to values near one, and that conclusion is confirmed here. Other comparisons are made with results of Lighthill and of Ostrach [NACA Tech. Note no. **3569** (1955)].

In an appendix to the paper, a study is made of the extension of two other, familiar, boundary-layer transformations to unsteady flows; these are, namely, the von Mises and Crocco transformations. The author concludes, tentatively, that the Howarth-Moore transformation seems simplest to apply in unsteady compressible boundary-layer theory. *W. R. Sears* (Ithaca, N.Y.).

v. Krzywoblocki, M. Z. On the boundary layer in a corner by use of the relaxation method. Ganita **7** (1956), 77–112.

The author investigates the flow of a viscous, compressible fluid in the corner between two semi-infinite planes at large Reynolds numbers. The method used is a direct generalization of that described by G. F. Carrier [Quart. Appl. Math. **4** (1947), 367–370; MR **8**, 415] for the incompressible case; this includes the use of the relaxation technique. As an example, the results of the calculation for the case of an insulated corner at Mach number 2 are given. *W. C. Rheinboldt* (Washington, D.C.).

Varley, E. An approximate boundary layer theory for semi-infinite cylinders of arbitrary cross-section. J. Fluid Mech. **3** (1958), 601–614.

The boundary-layer flow over a cylinder moving parallel to its generators is investigated by means of a Pohlhausen method of solution. The velocity distribution is chosen so as to yield a zero value for the terms representing the viscous retarding force in the boundary-layer equations. Since this assumption is known to be asymptotically valid far downstream, the method is expected to be most accurate there. However, even at the leading edge, where the Blasius solution applies, the method yields the correct square-root growth of the boundary layer, with a close approximation to the coefficient, for a large class of cross-sections. Results for the elliptic cylinder and the finite flat plate are given in closed form for the whole length of the cylinder, and an estimate is given of the effect of corners on skin frictional force and displacement area.

D. W. Dunn (Ottawa, Ont.).

Nazarchuk, M. M. On one method of solving the interior problem of boundary layer equations. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 145–148. (Ukrainian. Russian and English summaries)

The paper deals with an approximate method of solving compressible boundary-layer equations in the initial section of a plane canal.

The problem is reduced to integration of an ordinary differential equation for the form-parameter both in the case of a laminary layer and of a turbulent layer.

Author's summary.

Dem'yanov, Yu. A. Influence of a boundary layer on the character of the flow of a gas in a tube behind a moving shock wave. Prikl. Mat. Meh. **21** (1957), 473–477. (Russian)

Rozin, L. A. Some cases of similarity motions of an incompressible fluid in an unsteady laminar boundary layer. Prikl. Mat. Meh. **21** (1957), 361–367. (Russian)

Dem'yanov, Yu. A. Formation of a boundary layer on a plate with moving discontinuities of density. Prikl. Mat. Meh. **21** (1957), 368–374. (Russian)

Dolph, C. L.; and Lewis, D. C. On the application of infinite systems of ordinary differential equations to perturbations of plane Poiseuille flow. Quart. Appl. Math. **16** (1958), 97–110.

The authors first considered the time-dependent problem of the stability of channel flow in the linear approximation along the lines used by Hopf [Math. Nachr. **4** (1951), 213–231; MR **14**, 327] for the non-linear hydrodynamical equations. As far as the stability problem is concerned, this leads to the same boundary-value problem of the Orr-Sommerfeld equation in the usual stability theory. The authors carried out numerical calculations with the help of a large digital computer by using appropriate finite difference methods. The results are in only qualitative agreement with those obtained by this reviewer [The theory of hydrodynamic stability, Cambridge, 1955; MR **17**, 1022] using asymptotic methods and by L. H. Thomas [Phys. Rev. (2) **91** (1953), 780–783; MR **15**, 262] using machine computation. Nevertheless, instability is indicated. The authors also cited some isolated computations which are in better agreement with earlier results. [Reviewer's remark: The authors' opinion, emphasizing the nonlinear problem, is probably shared by all workers in this field; but their estimate of the current situation is highly controversial. For current literature on this subject of non-linear oscillations, see D.Meksyn and J. T. Stuart, Proc. Roy. Soc. London, Ser. A. **208** (1951), 517–526; MR **13**, 792; J. T. Stuart, paper reviewed below, and other references quoted therein.]

C. C. Lin (Cambridge, Mass.).

Stuart, J. T. On the non-linear mechanics of hydrodynamic stability. J. Fluid Mech. **4** (1958), 1–21.

Author attacks the important and difficult problem of a nonlinear disturbance in a basic laminar flow by emphasizing the fundamental process of energy transfer. In particular he considers the distortion of the mean flow due to the effect of the Reynolds stress and its consequence on the rate of energy transfer. It is suggested that, in many cases, an equilibrium state may be possible in which this rate of transfer of energy from the distorted mean flow to the disturbance balances precisely the rate of viscous dissipation of the energy of disturbance. For flow between rotating cylinders, the theory yields a definite value for the torque required to maintain the motion beyond the critical Taylor number. Good agreement is obtained with G. I. Taylor's measurements of the torque for the case when the inner cylinder rotates and the outer cylinder is at rest. *C. C. Lin.*

Kahan, T. Turbulence ionosphérique et propagation des ondes électromagnétiques. Nuovo Cimento (10) **4** (1956), supplemento, 1352–1384.

This is a review paper. The first twenty pages (§§ 1–8) are devoted to an account of the current ideas relating to turbulence: the notions of inertial transfer and the so-called universal range, Kolmogoroff's laws and Heisenberg's elementary theory. The last section (§ 9) is devoted to the scattering of electromagnetic radiation by a turbulent atmosphere. The treatment follows that of Villars and Weisskopf [Phys. Rev. (2) **94** (1954), 232–240; MR **15**, 762] and their result that the scattering cross-section is proportional to $(\sin\frac{1}{2}\theta)^{-13/3}$ (where θ is the scattering angle) is derived. *S. Chandrasekhar.*

Kraichnan, Robert H. Irreversible statistical mechanics of incompressible hydromagnetic turbulence. Phys. Rev. (2) **109** (1958), 1407–1422.

A theory of stationary, homogeneous turbulence is developed on the hypothesis that the primary contribution to the triple correlation of three distinct Fourier modes arises from their direct interaction. One of the results of the theory is that there is an appreciable coupling between the modes in the energy-containing and inertial ranges even for large values of the Reynolds number. For isotropic turbulence (in the absence of magnetic fields), this coupling leads to an energy distribution in the inertial range given by

$$E(k)=\text{constant}(\varepsilon v_0)^{\frac{1}{2}}k^{-3/2},$$

where ε is the rate of energy dissipation, v_0 is the root-mean-square value of the velocity fluctuations in one direction, and the constant can, in principle, be determined from the theory. This result differs from the familiar Kolmogoroff law based on the cascade hypothesis, in which the energy distribution is proportional to $\varepsilon^{2/3}k^{-5/3}$ and is independent of v_0. *W. H. Reid.*

Jain, P. C. On Chandrasekhar's theory of turbulence. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 504–513.

In the theory of stationary turbulence proposed by Chandrasekhar [Proc. Roy. Soc. London. Ser. A. **229** (1955), 1–19; MR **16**, 968], the principal mean value employed is the double correlation

$$Q_{ij}=\langle u_i(\mathbf{x}, t)u_j(\mathbf{x}+\mathbf{r}, t+\tau)\rangle_{\text{av}},$$

which is assumed to depend only on the spatial separation $\mathbf{r}$ (homogeneity) and the time difference τ (stationarity). In this theory it is further assumed that quadruple correlations are related to Q_{ij} as in a normal distribution. In the case of isotropic turbulence worked out by Chandrasekhar, one obtains a single dynamical equation for the defining scalar of Q_{ij}. The present author considers the more complicated case of stationary axisymmetric turbulence and succeeds in deriving the required eight scalar equations. *W. H. Reid* (Providence, R.I.).

Eckart, Gottfried. Über die Streuung des Schalles an Wirbeln und turbulenten Zonen in der Atmosphäre. Bayer. Akad. Wiss. Math.-Nat. Kl. Abh. (N.F.) no. **84** (1958), 27 pp.

The linearized equation of sound waves is set up for case of a medium which is a perfect inviscid gas in general motion. A first approximation is formulated for the perturbation of a plane sound wave arising when the motion of the medium is a known perturbation of a uniform flow. This is calculated in detail to obtain the diffraction of a plane sound wave incident axially on a particular stationary, isothermal, straight vortex of strength decaying rapidly with distance from a central point. *R. E. Meyer* (Providence, R.I.).

Sen, N. R. On decay of energy spectrum of isotropic turbulence. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 530–533.

Assuming the transfer of turbulent energy from Fourier components of small wave-number to components of larger wave-number to be described by a viscosity coefficient of the Heisenberg form, self-preserving decay of isotropic turbulence is possible with spectrum functions of form

$$F(k,t)=\frac{1}{K^2k_0^2t_0^2}\left(\frac{t_0}{t}\right)^{2-3c} f\left(\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c\right),$$

where c is a constant and $f(x)$ satisfies an integral equation. This form is only consistent with a variation of $F(k,t)$ as k^4 at low wave-numbers if $c=2/7$, although Heisenberg assumed that $c=\frac{1}{2}$. Using the numerical solutions of the equation defining $f(x)$, it is shown that the solution for $c=2/7$ is probably unstable for perturbations from the self-preserving form. On the other hand, C. C. Lin has shown that the solution for $c=\frac{1}{2}$ is probably stable and it is suggested that the Heisenberg spectrum ($c=\frac{1}{2}$) is attained through a series of labile stages represented by varying values of c. *A. A. Townsend.*

Binnie, A. M.; and Phillips, O. M. The mean velocity of slightly buoyant and heavy particles in turbulent flow in a pipe. J. Fluid Mech. **4** (1958), 87–96.

The first part of the paper gives an account of an experimental investigation of the problem. In the concluding section, a semi-empirical theory is developed to show how the mean velocity of the spheres depends upon their relative densities and sizes. *D. W. Dunn.*

Aksenov, A. P. The turbulent boundary layer on a cone in supersonic flow with account taken of heat radiation. Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **12** (1957), no. 19, 112–128. (Russian. English summary)

In this paper the problem of determination of surface temperature and skin friction on an insulated cone is solved. In the laminar sublayer the Prandtl number is assumed to be different from unity, and viscosity is assumed to vary according to Sutherland's formula. In the turbulent region the shear stress and the component of the heat-flow vector in the y-direction are assumed to be variable across the boundary layer.

Author's summary.

v. Krzywoblocki, M. Z.; and Hassan, H. A. On the limiting lines in diabatic flow. Comment. Math. Univ. St. Paul. **6** (1958), 115–139.

Cet article est consacré à l'étude de certains écoulements de fluides parfaits "diabatiques" que l'on peut considérer comme des écoulements dans lesquels des sources extérieures de chaleur peuvent être présentes. Il semble, en fait, que les auteurs — qui n'insistent guère sur les hypothèses faites, ni sur leur signification physique — se restreignent au cas où la chaleur totale introduite en un point est uniquement fonction de la vitesse. Dans ces conditions, la théorie classique de Chaplygin peut être généralisée. Deux problèmes sont ainsi envisagés, celui des lignes limites et celui de l'écoulement au col d'une tuyère. Notons que dans ces écoulements, la ligne sonique ne coïncide plus nécessairement avec la frontière des domaines elliptique et hyperbolique. *P. Germain.*

Shercliff, J. A. Some generalizations in steady one-dimensional gas dynamics. J. Fluid Mech. **3** (1958), 645–657.

Il s'agit d'une étude systématique des écoulements unidimensionnels stationnaires, dits "écoulements par tranches", généralisant la théorie classique des écoulements adiabatiques et les processus de Rayleigh et Fanno. Considérant comme grandeurs fondamentales $F=p+\rho u^2$, $G=\rho u$, $H=h+\frac{1}{2}\rho u^2$ et s, l'entropie spécifique, l'auteur examine en détail les différents cas où deux de ces grandeurs restent constantes, à la lumière des seuls principes de la thermodynamique, et donne une interprétation physique aux solutions ainsi étudiées. La signification et l'importance de la ligne sonique sont clairement mises en valeur. L'auteur fournit aussi une interprétation de la vitesse du son isotherme introduite par Newton.

P. Germain (Paris).

★**van de Vooren, A. I. Unsteady airfoil theory.** Advances in applied mechanics, Vol. V, pp. 35–89. Academic Press Inc., New York, N.Y., 1958. x+459 pp. $12.00.

This is a comprehensive survey of oscillating airfoil theory with approximately equal treatment of subsonic and supersonic flow; transient motions are treated briefly. Starting with a section devoted to the linearized equations of motion and their boundary conditions and transformations, there follow four sections on the oscillating airfoil in two- and three-dimensional, subsonic and supersonic flow, one on non-linear approximation, and one on indicial problems. The emphasis is on a clear but concise exposition of methods, rather than on the details of applications, and the result should be especially valuable for the non-specialist. There are 108 references. [For more detailed surveys, reference may be made to I. E. Garrick, Sect. F of "Aerodynamic components of aircraft at high speeds" [Princeton, 1957; MR **18**, 844] and J. W. Miles, "The potential theory of unsteady supersonic flow", Cambridge Univ. Press (in press, 1958). *J. W. Miles.*

Ryžov, O. S. On flows with degenerate hodograph. Prikl. Mat. Meh. **21** (1957), 564–568. (Russian)

Consider steady isentropic irrotational flows in which the rectangular velocity components satisfy $w=w(u,v)$, so-called double waves. The author derives the quasi-linear partial differential equation of second order satisfied by w. He shows that the characteristic curves on a hodograph surface $w=w(u,v)$ are related in a simple way to the hodograph characteristics of plane flow, and finds other geometrical analogies between plane flows and double waves. Similar results were previously found by A. A. Nikol'skiĭ [various Trudy Central. Aero-Gidrodinam. Inst. (1949)] and by the reviewer [Ballistic Res. Lab., Aberdeen Proving Ground, Md. Memo. Rep. No.

657 (1948); Quart. Appl. Math. **9** (1951), 237–246; MR **13**, 179]. The author also considers double waves in non-steady flows. *J. H. Giese* (Havre de Grace, Md.).

Nazarov, G. I. Solution of plane problems of gas dynamics by the method of multi-link approximation of Čaplygin's function. Tomskiĭ Gos. Univ. Uč. Zap. Mat. Meh. **25** (1955), 77–96. (Russian)

"A further development of the approximate methods of L. I. Sedov and S. Tomotika, and K. Tomada for solving plane gas dynamic problems is given herein on the basis of a multi-link approximation of the Čaplygin function, in the basic differential equation of barotropic gas motion, by similar curve sections. Conditions are established for the connection of the solution in the transition points from one hypothetical gas to another, which satisfy all the fundamental physical conditions of flow continuity at points where the streamlines blend. The Tomotika-Tomada inverse method to solve boundary problems is used for a multi-link hypothetical gas . . . A computation is made for a two and three-link hypothetical gas." (From the introduction.) *M. D. Friedman.*

Fal'kovič, S. V. On the theory of gas flow. Prikl. Mat. Meh. **21** (1957), 459–464. (Russian)

Yih, Chia-Shun. Maximum speed in steady subsonic flows. Quart. Appl. Math. **16** (1958), 178–180.

The author proves that for any subsonic, isotropic, irrotational, twice continuously differentiable flow of a perfect gas, the maximum speed occurs on the boundary. hTe method of proof is to show that there exists a positive definite form b_{ij} such that $b_{ij}(q^2)_{,ij} \geqq 0$. The author selects $b_{ij}=\delta_{ij}-\phi_{,i}\phi_{,j}/c^2$, which is obviously positive definite because at any point it may be diagonalized in the form $(1-M^2, 1, 1)$. *C. Truesdell* (Bloomington, Ind.).

Power, G.; and Smith, P. Compressible fluids in steady two-dimensional subsonic flow. Appl. Sci. Res. A. **7** (1958), 357–368.

A scheme is presented for generating approximate solutions of the hodograph equations of plane subsonic flow of a perfect gas. A first approximation is obtained by methods similar to Karman-Tsien but which do not yield singularities near the critical speed. This approximate solution is used to start an iteration for the exact solution but only one further step is presented or appears practical. The method is an inverse one as the flow boundary is altered in the iteration. The integrals which occur in the solution can be evaluated numerically.

J. J. Mahony (Sydney).

Robacker, John T.; and Wheelon, Albert D. A note on the solution of a non-linear transonic flow equation. J. Math. Phys. **35** (1956), 321–322.

Germain, P. Sur la détermination locale d'une aile optimum en régime supersonique. Rech. Aéro. no. **60** (1957), 3–12.

The theorems of R. T. Jones [J. Aero. Sci. **18** (1951), 75–81; **19** (1952), 813–822; MR **12**, 554; **14**, 511] regarding minimum-drag wings are re-derived. The extension announced by Nikolsky at the 9th International Congress for Applied Mechanics (Brussels, 1956), which relates the potential of the combined flow defined by Jones [loc. cit.] to a certain plane harmonic function, is demonstrated. To design a minimum-drag wing, one must solve an improperly set Cauchy problem. Following techniques of Hadamard, the general theory of this problem is worked out. For the optimum wing of given total lift, the pressure distributions along all transverse lines may be found in the form of series of Legendre polynomials. The general theory is applied to wings of elliptic planform, and the results of Jones [loc. cit.] are confirmed. Certain unanswered mathematical questions are pointed out. *W. R. Sears* (Ithaca, N.Y.).

★ **Li, Ting Yi. An inverse problem in hypersonic viscous flow.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 201–223. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

The paper gives an approximate method for integrating the boundary-layer equations, under the assumptions that the local pressure is related to the flow inclination by the tangent-wedge approximation and that the Prandtl number is equal to unity. For a special power-form pressure distribution, the problem is solved for three cases: (1) a heated surface, (2) a cooled surface, and (3) a supercooled surface. In each case, the body is found, except near the leading edge, to closely approximate a wedge.

Y. H. Kuo (Peking).

★ **Truitt, Robert W. Flow in the forward stagnation region of blunt bodies.** Proceedings of the Fifth Midwestern Conference on Fluid Mechanics, 1957, pp. 238–249. University of Michigan Press, Ann Arbor, Mich., 1957. viii+388 pp. $8.00.

The method presented in this paper is approximate. Its accuracy can be judged from the comparisons given in a review by Milton D. Van Dyke [J. of Aero. Sci. **25** (1958), 485–496]. *Y. H. Kuo* (Peking).

Ehlers, F. Edward; and Strand, Torstein. The flow of a supersonic jet in a supersonic stream at an angle of attack. J. Aero. Sci. **25** (1958), 497–506.

The author uses Laplace-transform theory to determine the shape of the boundary of a supersonic jet which is inclined at a small angle of incidence in a supersonic free stream. Two- and three-dimensional jets are considered. It is shown that, in both these cases, the shape of the jet boundary depends on the parameter

$$k=(\gamma_1 P_1 M_1^2/\gamma_2 P_2 M_2^2)[(M_2^2-1)/(M_1^2-1)]^{\frac{1}{2}},$$

where P is the pressure, M is the Mach number and the subscripts 1 and 2 refer to the internal and external flows, respectively. The behaviour of the jet depends upon whether k is less than, greater than or equal to one. A further interesting result is that, in the three-dimensional case, the shape of the jet boundary depends on the length of the tube which produces the jet; this is not true for two-dimensional jets. *G. N. Lance* (Southampton).

Guiraud, Jean-Pierre. Ecoulements hypersoniques infiniment voisins de l'écoulement sur un dièdre. C. R. Acad. Sci. Paris **244** (1957), 2281–2284.

Dans le cadre de la théorie hypersonique des petites perturbations, on étudie l'écoulement sur un profil quelconque assujetti à la seule condition d'être voisin d'un dièdre. Le résultat obtenu contient comme cas particulier la formule de Van Dyke pour le gradient de pression à la pointe d'une ogive et il est en accord avec l'approximation newtonienne. *Résumé de l'auteur.*

Fenain, Maurice; et Vallée, Denise. Effets de portance, en régime supersonique, pour certaines ailes en flèche effilées. Calcul de traînée minimum. C. R. Acad. Sci. Paris **244** (1957), 1138–1141.

L'application de la théorie des mouvements homogènes à l'étude de certaines ailes en supersonique fait l'objet d'une formulation systématique, permettant de calculer aisément les effets de portance et se prêtant bien à la recherche d'optimum. Les résultats d'applications faites dans ce sens sont comparés à l'optimum optimorum obténu par P. Germain. *Résumé de l'auteur.*

D'iakov, S. P. The interaction of shock waves with small perturbations. I, II. Soviet Physics JETP **6** (1958), 729–747.

In this posthumous paper the perturbation of a plane shock in steady two-dimensional flow of inviscid gas due to a given small perturbation of the uniform flow on either side is calculated on a strictly linearized approximation. Part I treats the case of flow supersonic on both sides of the shock, and it is pointed out that the linearized approximation breaks down at a certain Mach number of the incident flow. It seems unlikely that any of the results are new, but in view of the completeness of the formulae, they might form a useful appendix to surveys like R. Courant and K. Friedrichs, Supersonic flow and shock waves, Interscience, New York, 1948 [MR **10**, 637], or H. Polachek and R. J. Seeger, Shock wave interactions, in "Fundamentals of gas dynamics," Princeton Univ. Press, Princeton, 1958, pp. 482–525. Part II treats the case of subsonic flow downstream of the shock, and the linearized approximation always breaks down inasmuch as it predicts singularities; when the incident perturbation possesses a discontinuity of derivatives, the singularity is logarithmic. *R. E. Meyer* (Providence, R.I.).

Raizer, Iu. P. On the structure of the front of strong shock waves in gases. Soviet Physics. JETP **5** (1957), 1242–1248.

L'auteur étudie la structure d'une onde de choc en tenant compte de l'énergie dégagée par rayonnement. Les profils classiques: variation des grandeurs aérodynamiques et de l'intensité du rayonnement sont construits.

H. Cabannes (Marseille).

Hristianovič, S. A. Shock wave at considerable distance from the place of explosion. Prikl. Mat. Meh. **20** (1956), 599–605. (Russian)

The author writes the equations for unsteady isentropic spherically symmetric flow in terms of independent variables $\tau=\ln t$ and $\beta=r/t$. He first considers water with $p=B[(\rho/\rho_0)^n-1]$ and approximates $a=(dp/d\rho)^{0.5}$ and ρ/ρ_0 by expressions of the form $A+Cp/Bn$. He also writes $\beta=a_0(1+\delta)$, $a=a_0(1+\alpha)$, $u=a_0M$, where u is the fluid velocity, and δ, α, and M are assumed small. In the equation of continuity the author retains only first order terms. In the equation of motion he also retains some second order terms. Solutions of the resulting system that are independent of τ can be found explicitly to yield, eventually, $r/t=\beta(p)$. By calculating shock velocity $N=dr/dt$ from this relation and comparing it with the approximation $N=a_0[1+(n+1)p/4Bn]$, the author obtains an integrable equation $dt/dp=f(p)$, from which $t(p)$ and then shock radius $r(p)$ can be found. The results, independent of the initial conditions of the explosion, coincide with previous shock attenuation estimates of L. D. Landau [Prikl. Mat. Meh. **9** (1945), 286–292; MR **8**, 111] and agree well with experimental data. Cylindrical and plane shocks are treated similarly. Gases are also discussed, with the obvious change $\Delta p=p_0[(\rho/\rho_0)^\gamma-1]$. The procedure just described is limited to $p>0$ for water and $\Delta p>0$ for gases. For gases, regions of flow with $\Delta p<0$ are treated by retaining in the equation of motion the term $\partial\alpha/\partial\tau$. This yields a first order quasi-linear partial differential equation for $M(\delta,\tau)$, which can be solved explicitly. This solution involves a term $t_0t^{-1}\delta_0(Mt/t_0)$, which depends on the initial Mach number distribution $\delta_0(M)$ in the flow at time t_0. *J. H. Giese.*

Thomas, T. Y. On the propagation and decay of spherical blast waves. J. Math. Mech. **6** (1957), 607–619.

If the radius $R(t)$ of a spherical blast wave propagated into a quiescent atmosphere were known, the flow inside would be completely determined. The author proposes to find $R(t)$ approximately as follows. Let p, ρ, v, E, and c be pressure, density, gas velocity, energy per unit mass, and speed of sound. Let subcripts 0 denote undisturbed values; and bars, values on the rear side of the shock. From familiar shock conditions, $[E]=\bar{E}-E_0=\bar{p}\bar{v}/\rho_0G$, where $G=dR/dt$. Also, at time $t+\Delta t$, for the fluid in the spherical shell bounded by $R(t)$ and $R(t+\Delta t)=R(t)+\Delta R$, $4\pi R^2\bar{\rho}\Delta R(E-E_0)=\Delta Q$ is the contribution of the original energy Q of the explosion. The author assumes $\Delta Q=\alpha Q\Delta V/V(t)$, where $V(t)=4\pi R^3/3$, $\Delta V=4\pi R^2\Delta R$, and α is a constant. Then (*) $[E]=3\alpha Q/4\pi R^3\bar{\rho}=\bar{p}\bar{v}/\rho_0G$ yields a first order ordinary differential equation for $R(t)$. For a point explosion $\alpha=(\gamma+1)/(\gamma-1)$. For $R\to0$, (*) implies $t=O(R^{2.5})$. For $R\to\infty$, (*) implies $G\sim c_0$ and $\bar{p}\sim p_0$.

J. H. Giese (Havre de Grace, Md.).

Thomas, T. Y. The growth and decay of sonic discontinuities in ideal gases. J. Math. Mech. **6** (1957), 455–469.

Consider a non-steady three-dimensional compressible flow. Let $\Sigma(t)$ be a moving characteristic surface on which velocity components v_i, pressure p, and density ρ are continuous, but some of their first partial derivatives actually have discontinuities, with jumps $[\partial v_i/\partial x_j]$, etc. Let ν_i be the unit normal at a point P of $\Sigma(t)$. Then at P the jumps $[\partial v_i/\partial x_j]=\lambda\nu_i\nu_j$, $[\partial p/\partial x_i]=\xi\nu_i$, $[\partial\rho/\partial x_i]=\zeta\nu_i$, etc., are characterized by the scalars λ, ξ, and ζ. Now suppose $\Sigma(t)$ advances into homogeneous stagnant gas. By very efficient and elegant applications of tensor methods and differential geometry, the author obtains the following results. The orthogonal trajectories of the family $\Sigma(t)$ are straight lines; in fact, $\Sigma(t)$ is a family of parallel surfaces. The values of $\lambda(t)$, $\xi(t)$, and $\zeta(t)$ on an orthogonal trajectory are determined explicitly as functions of t, $\lambda(0)$, $\xi(0)$, $\zeta(0)$, and the mean and total curvatures Ω_0 and K_0 of $\Sigma(0)$. In terms of various combinations of signs of $\lambda(0)$, Ω_0, and K_0, the author completely analyzes all possibilities for attenuation ($\lambda\to0$) or amplification ($|\lambda|\to\infty$) of the jumps $[\partial v_i/\partial x_j]$, etc.

J. H. Giese (Havre de Grace, Md.).

Grib, A. A.; Ryabinin, A. G.; and Hristianovič, S. A. On the reflection of a plane shock wave in water from a free surface. Prikl. Mat. Meh. **20** (1956), 532–544. (Russian)

Pekeris, Chaim L.; and Longman, Ivor M. Ray-theory solution of the problem of propagation of explosive sound in a layered liquid. J. Acoust. Soc. Amer. **30** (1958), 321–328.

This is a reexamination of a problem already solved by

Pekeris [Geol. Soc. Amer. Mem. **27** (1948)], using a different mathematical approach. The problem is that of a transient pressure point source (a step function $H(t)$) in a liquid layer of thickness H overlying a fluid half space (the "bottom"). The formulation is operational, and has been discussed elsewhere by Pekeris [Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 439–443; MR **17**, 1216], Cagniard [Réflexion et réfraction des ondes seismiques progressives, Gauthier-Villars, Paris, 1939] and others. The operational form can be represented as an infinite series, of which each term describes a separate arrival with definite time of commencement. These terms are interpreted and evaluated numerically on an electronic computer. From the physical standpoint, the results show three interesting features.

First, it is shown that for a high speed bottom, if the range of observation is less than that necessary for the appearance of totally reflected rays, there is no essential difference from the low speed bottom solution.

Second, corresponding to the critical rays (i.e., the "refracted arrivals") there is a logarithmically infinite arrival, and it is pointed out by Pekeris that this infinity is due to the discontinuous form of the pulse — i.e., a sharply peaked but continuous variation of pressure at the source would give finite, although large, signals at these points.

Third, although the number of arrivals to be summed becomes very large at long ranges, Pekeris states that the first two thirds cancel each other out, and that the physically measurable results are given by the last third. However, it seems probable to the reviewer that this result is special to the problem considered, and hinges upon the type of pulse, the bottom, and the source and receiver depths. Thus, situations can be imagined where this would not be so — e.g., in the case of $\delta(t)$ pressure point source in a perfect wave guide, all arrivals would be distinct for most source and receiver depth combinations.

I. Tolstoy (Dobbs Ferry, N.Y.).

Goldberg, Z. A. On the propagation of plane waves of finite amplitude. Akust. Ž. **3** (1957), 322–328. (Russian)

This paper presumes acquaintance with the author's earlier work [same Ž. **2** (1956), 325–328; **3** (1957), 149–153; MR **19**, 1008], in which results of C. Eckart [Phys. Rev. (2) **73** (1948), 68–76] on second-order acoustic effects are extended to additional cases of sound propagation. In the present paper a numerical criterion in terms of the propagation parameters is given for the appearance of discontinuities in the wave form, and an estimate is made for the distance at which such a discontinuity develops. The increase in the absorption coefficient because of motion of the medium is calculated for the respective cases of discontinuity formation and no discontinuity formation. The values obtained are compared with known theoretical and experimental results, and are found to be in reasonable agreement.

R. N. Goss (San Diego, Calif.).

Jones, D. S.; and Whitham, G. B. An approximate treatment of high-frequency scattering. Proc. Cambridge Philos. Soc. **53** (1957), 691–701.

By using arguments of the type which Fresnel and Kirchoff used in diffraction theory, the authors deduce the results recently derived by Kodis [Trans. I.R.E., AP-4 (1956), 580]. The work of Kodis is concerned with the scattering of a plane harmonic wave by a circular cylinder for the case of high frequency waves. The authors discuss some two and three dimensional obstacles.

A. E. Heins (Pittsburgh, Pa.).

Narasimhan, M. N. L. Acoustic streaming in liquids contained in elastic tubes. J. Assoc. Appl. Phys. Calcutta Univ. **4** (1957), 39–52.

Ericksen, J. L. Characteristic direction for equations of motion of non-Newtonian fluids. Pacific J. Math. **7** (1957), 1557–1562.

L'Auteur établit plusieurs résultats concernant l'existence des directions caractéristiques dans la théorie de Reiner-Rivlin, du mouvement d'un fluide non newtonien. Le tenseur des tensions étant

$$t_j^i=-p\delta_j^i+F_1d_j^i+F_2d_k^id_j^k,$$

il forme, en particulier, une condition simple qui, dans le cas où $F_2\equiv 0$, est nécessaire et suffisante pour qu'il n'existe pas de directions caractéristiques réelles.

R. Gerber (Grenoble).

Piddington, J. H. The motion of ionized gas in combined magnetic, electric and mechanical fields of force. Monthly Not. Roy. Astr. Soc. **114** (1954), 651–663 (1955).

Let a constant magnetic field of strength H (in the z-direction, say) pervade a uniform ionized gas. At time $t=0$, apply a transverse external electric field (with components E_x and E_y) and a mechanical force F (along the x-direction). The motion of the gas which follows will be in the transverse plane. If v_x and v_y are the components of the velocity, the electric field $\boldsymbol{E}'$ experienced by the gas has the components E_x+v_yH and E_y-v_xH. According to the equation

$$\boldsymbol{j}=\sigma_1\boldsymbol{E}_\perp'+\sigma_2\boldsymbol{H}\times\boldsymbol{E}'_\perp/H \tag{1}$$

derived by the author [same Not. **114** (1954), 638–650; MR **17**, 691; the notation is the same as in that review], the induced current has the components

$$\begin{aligned} j_x&=\sigma_1(E_x+v_yH)-\sigma_2(E_y-v_xH),\\ j_y&=\sigma_1(E_y-v_xH)+\sigma_2(E_x+v_yH). \end{aligned} \tag{2}$$

Under the influence of these currents and the force F, the gas is accelerated in accordance with the equation of motion

$$\rho\frac{d\boldsymbol{v}}{dt}=\boldsymbol{j}\times\boldsymbol{H}+\boldsymbol{F}\quad(\rho=\text{density}). \tag{3}$$

With $\boldsymbol{j}$ given by (2), this equation can be integrated. The solution is

$$\begin{aligned} v_x&=A(1-\varepsilon^{-\alpha t}\cos\beta t)+B\varepsilon^{-\alpha t}\sin\beta t,\\ v_y&=-B(1-\varepsilon^{-\alpha t}\cos\beta t)+A\varepsilon^{-\alpha t}\sin\beta t, \end{aligned} \tag{4}$$

where $A=E_y/H+F/H^2\sigma_3$, $B=E_x/H+F\sigma_2/H^2\sigma_1\sigma_3$, $\sigma_3=\sigma_1+\sigma_2^2/\sigma_1$, $\alpha=H^2\sigma_1/\rho$ and $\beta=H^2\sigma_2/\rho$. The solutions for $\boldsymbol{j}$ and $\boldsymbol{E}'$ follow from (4). The author discusses this solution under a variety of astrophysical conditions and indicates the circumstances under which the solution will describe the physical situations consistently.

S. Chandrasekhar (Williams Bay, Wis.).

Khalatnikov, I. M. On magnetohydrodynamic waves and magnetic tangential discontinuities in relativistic hydrodynamics. Soviet Physics. JETP **5** (1957), 901–905.

The Rankine-Hugoniot equations describing the conditions at a plane shock front in relativistic magneto-

hydrodynamics are obtained by writing the total energy-momentum tensor of the matter and the field and requiring that its components normal to the front are continuous. The velocity of propagation of waves of infinitesimal amplitude is obtained by passing to the limit of zero discontinuity in the shock equations. [The results are fairly obvious generalizations of those given in an early paper of de Hoffmann and Teller, Phys. Rev. (2) **80** (1950), 692–703; MR **12**, 769.]

S. Chandrasekhar (Williams Bay, Wis.).

Agostinelli, Cataldo. Moti magneto idrodinamici simmetrici rispetto a un asse. Caso delle piccole oscillazioni in una massa fluida sferoidale. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **91** (1956–57), 263–298.

The equations governing axisymmetric magnetic fields and fluid motions in an incompressible conducting viscous fluid are reduced by introducing stream functions to describe the poloidal parts of the field. Solutions of these equations are sought for the following two cases: (1) a uniformly rotating fluid and (2) the oscillations of a slowly rotating spheroid. {It seems to the reviewer that the basic equations can be expressed more conveniently in terms of the scalars introduced by Lüst and Schlüter [Z. Astrophys. **34** (1954), 263–282; MR **17**, 110]. Equations equivalent to those given by the author will be found in S. Chandrasekhar, Astrophys. J. **124** (1956), 232–243 [MR **18**, 86]. Again, insufficient attention has been paid to the boundary conditions: for example, the solution for case (1) involves Bessel functions for a purely imaginary argument which diverge at infinity. Also, in case (2) the assumptions that the configuration is self-gravitating and that a uniform magnetic field pervades all space are not consistent. Nevertheless, the author's attempt is a valiant one on difficult problems.} *S. Chandrasekhar.*

Resler, E. L., Jr.; and Sears, W. R. The prospects for magneto-aerodynamics. J. Aero. Sci. **25** (1958), 235–245, 258.

Les équations de l'écoulement d'un fluide électriquement conducteur sont écrites en adjoignant aux équations de conservation du moment, de l'énergie et de la masse les équations de Maxwell et la loi d'Ohm. Sous certaines hypothèses simplificatrices, la présence de courants électriques se traduit par l'introduction dans les deux premières équations respectivement d'une densité de force $\mathbf{j}\times\mathbf{h}$ et de l'effet Joule j^2/σ.

Les effects électromagnétiques deviennent assez grands pour affecter les écoulements à haute altitude et aux grandes vitesses, et plus particulièrement si la conductivité électrique σ peut être augmentée et stabilisée par l'introduction dans l'air d'une substance facilement ionisable [R. J. Rosa, Thesis, Cornell Univ., 1956]. L'étude du nombre de Reynolds magnétique Rm permet de justifier l'hypothèse "conductivité électrique infinie" souvent employée pour la résolution des problèmes astrophysiques. En aéronautique cette hypothèse ne semble pas appropriée, et les conclusions apparaissent moins nettement.

Des exemples illustrent certains effets magnéto-aérodynamiques tels que la décélération de l'écoulement et sa possible application au problème de la ré-entrée. Des résultats obtenus par Rossow [NACA Tech. Note no. **3971** (1957)] sont rappelés pour deux types d'écoulements de Poiseuille, entre des murs isolants ou non, relativement au profil des vitesses et à la répartition énergetique.

Dans le cas stationnaire quasi-unidimensionnel les auteurs obtiennent l'expression de l'accélération et dressent un tableau des différents types possibles d'écoulements en fonction du nombre de Mach. Les résultats diffèrent largement de ceux de la dynamique des gaz. On peut par exemple obtenir dans une tuyère à section constante un écoulement supersonique accéléré, et même rendre cette accélération maximum en choisissant convenablement le rapport E/H (E, H intensités des champs électrique et magnétique). En effet les auteurs supposent que les champs ne sont pas affectés par les variations de vitesse, et seraient par conséquent des fonctions dont on peut disposer (les équations de Maxwell ne sont pas écrites), ce qui offrirait d'attrayantes possibilités de contrôle de l'accélération, de la décélération, etc., dans les limites imposées par l'existence de conditions et de difficultés d'origine physique — potentiel de rupture, fortes résistances à la surface des électrodes. *J. Naze.*

Shafranov, V. D. The structure of shock waves in a plasma. Soviet Physics. JETP **5** (1957), 1183–1188.

La structure d'une onde de choc dans un plasma (gas ionisé) est étudiée en supposant que les électrons et les ions peuvent être à des températures différentes. Un système d'équations différentielles déterminant les variations de température est formé et intégré numériquement. En l'absence de champ magnétique, les températures des ions et des électrons supposées initialement égales augmentent avant le choc; sur l'onde de choc la température des ions est discontinue tandis que la température des électrons est continue; après le choc les deux températures tendent vers une même valeur limite. La présence d'un champ magnétique parallèle au plan du choc a pour effet de diminuer la conductivité thermique des électrons et de supprimer la région d'échauffement avant le choc.

H. Cabannes (Marseille).

Shafranov, V. D. On magnetohydrodynamical equilibrium configurations. Soviet Physics JETP **6** (1958), 545–554.

L'auteur étudie quelques configurations d'équilibre pour un courant électrique plaçé dans un milieu conducteur. Trois cas sont envisagés; des anneaux de courant (analogues à des tores) sont en équilibre, dans le premier cas sous l'action de forces de gravitation, dans le second cas sous l'action d'une pression extérieure, dans le troisième cas sous l'action d'une pression et d'un champ magnétique extérieurs. Dans la dernière partie, l'analogie entre ces équilibres et les tourbillons hydrodynamiques est étudiée; cette analogie permet de trouver une nouvelle configuration d'équilibre équivalente au tourbillon de Hill. *H. Cabannes* (Marseille).

Kihara, Taro. Macroscopic foundation of plasma dynamics. J. Phys. Soc. Japan **13** (1958), 473–481.

Les équations de la magnéto-aérodynamique sont établies dans le cas d'un fluide comprenant plusieurs composants; le cas de deux composants (ions, électrons) correspond aux gaz totalement ionisés. Les phénomènes quasi-stationnaires (correspondant aux valeurs négligeables des dérivées par rapport au temps) sont étudiés et il est établi que la loi d'Ohm d'après laquelle le vecteur $\sigma^{-1}\mathbf{J}-(\mathbf{E}+\mathbf{v}\times\mathbf{B})$ est nul peut être remplacée par la condition que ce vecteur soit le gradient d'une fonction arbitraire. *H. Cabannes* (Marseille).

Ohkawa, Tihiro; and Kihara, Taro. Externally pinched plasma. J. Phys. Soc. Japan **13** (1958), 482–483.

Les équations établies dans l'article précédent sont appliquées à l'étude de la stabilité d'une colonne de plasma excitée par un champ magnétique extérieur parallèle à la colonne. Le cas où l'indice adiabatique possède la valeur $\gamma=2$ conduit à une solution particulièrement simple.

H. Cabannes (Marseille).

Oroveanu, T. Sur quelques problèmes d'écoulement à travers un milieu poreux inhomogène. Rev. Méc. Appl. **1** (1956), no. 2, 83–91.

The author obtains certain solutions related to the problem of the flow of incompressible fluids in inhomogeneously porous media. Only plane flows are considered. The present type of inhomogeneity is characterized by variable permeability. This permeability, as a function of the co-ordinates r, θ, is represented in the form $f_1(r)/f_2(\theta)$. This type of dependence reduces the problem to the study of a Fuchsian class of differential equations. However, the author does not seem to be aware of the fact that these types of permeability functions have been studied previously [cf., e.g., Yalin, Bull. Tech. Univ. Istanbul **7** (1954), 59–78; MR **17**, 1151]. It is also surprising that the author does not make a single reference to any of the fundamental contributions of Polubarinova-Kočina [cf. Theory of motion of ground water, Gostehizdat, Moscow, 1952; MR **17**, 1151]. As is well known, the application of the Fuchsian theory is a central theme in some of her works.

K. Bhagwandin (Oslo).

Heinrich, G.; und Desoyer, K. Praktische Methoden zur Lösung von Problemen der stationären und instationären Grundwasserströmungen. Ing.-Arch. **26** (1958), 30–42.

In the present paper the authors continue their investigations [Ing.-Arch. **23** (1955), 73–84; **24** (1956), 81–84; MR **17**, 205; **18**, 91] related to stationary and non-stationary ground-water flow. The authors present numerical results based on finite difference methods. Relaxational techniques are employed. Unfortunately, the authors do not seem to be familiar with the vast modern literature concerning ground-water flow (cf. previous review).

K. Bhagwandin (Oslo).

Golovan', V. M. On filtration from a trapezoidal section channel, taking into account soil capillarity. Akad. Nauk Ukrain. RSR. Prikl. Meh. **3** (1957), 451–459. (Ukrainian. Russian and English summaries)

See also: Special Functions: Palm. **Partial Differential Equations:** Finn and Gilbarg. **Elasticity, Plasticity:** Berry and Reissner. **Quantum Mechanics:** Halbwachs. **Relativity:** Unal and Vigier. **Astronomy:** Pickelner; Kaplan; Jaggi. **Geophysics:** Krylov; Pierson.

Optics, Electromagnetic Theory, Circuits

Hänsel, Horst. Über die Berechnung der Seidelschen Bildfehlerausdrücke. Wiss. Z. Pädagog. Hochsch. Potsdam. Math.-Nat. Reihe **3** (1956/1957), 57–61.

It is shown that the expressions for the third order aberrations in J. Picht, Grundlagen der geometrisch-optischen Abbildung [Deutsch. Verlag Wiss., Berlin, 1955] are equivalent to the conventional Schwarzschild-Kohlschütter formulation of the Seidel aberration coefficients. There are many misprints. A factor $\sin 2\Psi_1$ is missing from the second term on the right of the second equation in (1); the subscript 1 is missing from z_1 in two places in (2). {In the notation of the paper the symbols Ψ and z are embellished by suprapositioned circles that we do not challenge the printer to reproduce here.} A summation sign is missing in (2b) and n_1^3 in (5e) should read n_1.

G. L. Walker (Southbridge, Mass.).

Pargamanik, L. E. On the kinetic theory of an electron gas in the presence of boundaries. Soviet Physics. JETP **6** (1958), 194–197.

The kinetic equations of an electron gas in a plane diode are used to show how one can construct a distribution function satisfying given conditions of emission and reflection at the electrodes for the one-dimensional case, neglecting the collision integral. Corrections to Langmuir's result, important for small anode potentials, are obtained.

Author's summary.

Edwards, S. F. A variational calculation of the equilibrium properties of a classical plasma. Phil. Mag. (8) **3** (1958), 119–124.

Körper, K. Schwingung eines Plasmazylinders in einem äusseren Magnetfeld. Z. Naturf. **12a** (1957), 815–821.

Ziel der Arbeit ist die in einem unendlich langen kreiszylindrischen Plasmazylinder — in dem entlang der Achse ein homogenes statisches Magnetfeld $\mathfrak{H}_0$ wirksam ist — auftretenden elektromagnetischen Schwingungen zu untersuchen, welche von einem ebenfalls unendlich langen den Kreiszylinder umgebenden Solenoid angeregt werden; zwischen beiden befindet sich ein endlicher Zwischenraum. Die Wärmeleitfähigkeit und die Viskosität werden vernachlässigt und ausserdem wird angenommen, dass $|\mathfrak{H}|\ll|\mathfrak{H}_0|$ ist, wo $\mathfrak{H}$ die magnetische Feldintensität des elektromagnetischen Wechselfeldes bedeutet. Den Ausgangspunkt der Berechnungen im Plasma bilden erstens die zwei Maxwellschen Gleichungen

$$(1)\qquad \operatorname{rot}\mathfrak{H}=\frac{4\pi}{c}\mathfrak{j}+\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{E} \text{ und } \operatorname{rot}\mathfrak{E}=-\frac{1}{c}\frac{\partial}{\partial t}\mathfrak{H},$$

weiter die Bewegungsgleichung

$$(2)\qquad \rho\frac{\partial v}{\partial t}=\frac{1}{c}[\mathfrak{j}\times\mathfrak{H}_0]$$

und das verallgemeinerte Ohmsche Gesetz

$$(3)\qquad \frac{4\pi}{\omega_p^2}\left(\frac{\partial\mathfrak{j}}{\partial t}+\gamma\mathfrak{j}\right)=\mathfrak{E}+\frac{1}{c}[v\times\mathfrak{H}_0]-\frac{m_i-m_e}{e}\frac{\partial v}{\partial t}.$$

γ bedeutet die Stossfrequenz zwischen Ionen und Elektronen, $\mathfrak{j}$ die Stromdichte, m_i die Masse der Ionen und m_e die der Elektronen, ausserdem ist ω_p die Plasmafrequenz. ($\omega_p^2=4\pi\rho e^2/m_i m_e$.) Alle anderen Symbole haben die gewohnte Bedeutung. Im Zwischenraum gelten nur die Maxwellschen Gleichungen (1) mit $\mathfrak{j}=0$ und im Solenoid mit $\mathfrak{j}=\text{konst}\neq 0$. Es wird ein dem Problem angepasstes Koordinatensystem (z, r, φ) eingeführt und dann werden nach der gewohnten Methode aus (1), (2) und (3) die Grenzbedingungen hergeleitet. Unter der Annahme einer rein periodischen Zeitabhängigkeit ($e^{+i\omega t}$) und Einführung des erwähnten Koordinatensystems erhält man aus (1), (2) und (3) 12 Gleichungen, die jedoch in zwei Systeme zerfallen. Aus dem ersten folgt eine Besselsche Differentialgleichung für E_φ und aus dem zweiten eine solche für

H_φ, H_φ, E_z und j_z werden deshalb durch allgemeine Zylinderfunktionen ausgedrückt; diesen Schwingungstyp kann jedoch die Spule nicht anregen. Ähnlich werden E_φ, E_r, H_z, j_φ, j_r, v_φ und v_r durch Besselsche Funktionen erster Art und erster und nullter Ordnung dargestellt. Alles das bezieht sich auf das Plasma. Im inneren Vakuumring werden E_φ und H_z durch Linearkombinationen von Besselschen und Neumannschen Funktionen beschrieben und ausserhalb der Spule werden diese Vektoren durch Hankelsche Funktionen zweiter Art asymptotisch dargestellt. Wenn man j_φ innerhalb des Solenoids als konstant betrachtet, so treten in der allgemeinen Lösung im Ausserraum auch noch Lommelsche Funktionen auf. Der Brechungsindex des Plasmas und die Energiegleichung werden hergeleitet. *T. Neugebauer.*

Schlüter, A. Der Gyro-Relaxations-Effekt. Z. Naturf. **12a** (1957), 822–825.

Bekannterweise wäre es zum Zwecke der technischen Energiegewinnung sehr vorteilhaft, wenn man dazu die bei der zwischen zwei Deuteronen auftretende Kernreaktion frei werdende sehr grosse Energie ausnützen könnte. Die Erzeugung von elektrischen Entladungen sehr grosser Energiedichte ist deshalb ein sehr wichtiges Problem. Die Stossquerschnitte in einem vollständig ionisierten Plasma sind jedoch zu T^{-2} proportional und es ist auch schon deshalb sehr schwierig einem Plasma durch Joulesche Wärme eine genügend grosse Energie zuzuführen. In der vorliegenden Arbeit wird gezeigt, dass es eine bis jetzt noch nicht beachtete theoretische Möglichkeit gibt, auf solch einem Plasma Energie zu übertragen.

Der Verfasser betrachtet geladene Teilchen in einem homogenen Magnetfeld H, dass sich zeitlich so langsam ändert, dass

$$(1) \qquad |(1/H)(dH/dt)| \ll \omega_g$$

ist, wo $\omega_g = |eH|/mc$ die Gyrationsfrequenz (Larmorfrequenz) bedeutet. Es bezeichne weiter E_p die kinetische Energie entlang der magnetischen Kraftlinien und E_q darauf senkrecht. (Im Gleichgewicht ist dann $E_q = 2E_p$.) Aus der Gleichung für das von dem Umlauf des geladenen Teilchens erzeugte Moment μ ($\mu = E_q/H$) und der Annahme, dass bei einer kleinen Störung des Gleichgewichtes

$$(2) \qquad \frac{d}{dt}\,(E_q - 2E_p) = -\gamma(E_q - 2E_p) \text{ für } \gamma > 0$$

gesetzt werden kann, folgt schliesslich die Differentialgleichung

$$(3) \qquad \frac{d}{dt}\frac{E_q^2 E_p}{H^3} = \frac{\gamma}{3}\frac{E_q(2E_p - E_q)^2}{H^3}.$$

Da die rechte Seite stets positiv ist, so muss der Quotient auf der linken Seite stets zunehmen. Wenn sich also das magnetische Feld auch periodisch ändert, so muss $E = E_p + E_q$ während einer ganzen Periode sicher zunehmen. (3) wird für den Fall $H(t) = H_0(1 + A\cos\omega t)$ weiter besprochen. Die zwei Fälle $\gamma \gg \omega$ und $\gamma \ll \omega$ werden gesondert betrachtet. Weiter werden die Bedingungen angegeben, unter denen die Ionen nach dieser Methode (gegenüber den Elektronen) bevorzugt aufgeheizt werden. *T. Neugebauer* (Budapest).

Wait, James R. The transient behavior of the electromagnetic ground wave on a spherical earth. Trans. I.R.E. **AP-5** (1957), 198–202.

The transient response to a current pulse at a source located above a conducting sphere is calculated. The result is first expressed formally as a Fourier integral in terms of the frequency spectrum of a general source. It is then particularized by choosing as a source one whose moment (instantaneous height times instantaneous current) is a "ramp" function. Numerical results for various conditions of propagation near the surface of the earth are obtained and discussed. *R. N. Goss.*

Busbridge, I. W. Finite atmospheres with isotropic scattering. I. Monthly Not. Roy. Astr. Soc. **115** (1955), 521–541 (1956).

Some transfer problems in plane parallel atmospheres of finite optical thickness τ_1 are considered, the scattering being isotropic with albedo $\bar{\omega}$ ($0 < \bar{\omega} \leqq 1$). The equations for Chandrasekhar's X- and Y-functions are obtained by a simple mathematical method, an extension of one originally due to Ambartsumian. The same method is then used to obtain expressions for the emergent intensities at both faces of the atmosphere when there is thermal emission and when it is subject to incident radiation. (Author's summary.) *A. Erdélyi.*

Busbridge, I. W. Finite atmospheres with isotropic scattering. II. Increase of line strength to the limb. Monthly Not. Roy. Astr. Soc. **116** (1956), 304–313 (1957).

The following problem in the theory of the formation of absorption lines is solved by the method developed in Paper I. An atmosphere of finite thickness lies above a photosphere of infinite extent, both being stratified in plane parallel layers. The absorption lines are formed by coherent isotropic scattering in the atmosphere, and there is continuous emission according to a linear Planck function throughout both photosphere and atmosphere. It is required to find the emergent intensity at any point of the line profile. Subject to certain assumptions, the exact solution is found in terms of the X- and Y-functions and their moments. Some numerical results are obtained and a theory is put forward to account for the variations in the intensities of faint and strong lines across the solar disk. (Author's summary.) *A. Erdélyi.*

Busbridge, I. W. Finite atmospheres with isotropic scattering. III. Corrigendum and addendum. Monthly Not. Roy. Astr. Soc. **117** (1957), 516–520.

The conclusions of the Appendix of Paper I are wrong. A rigorous treatment has established the correctness of all the main formulae and it has led to some new and important relations. These are summarized, and they are used to re-establish any results depending on the Appendix. (Author's summary.) *A. Erdélyi.*

Nedviga, P. Ya. Some problems of interference and diffraction of parallel pencils of light rays. Zaporiz. Derž. Ped. Inst. Nauk. Zap. Fiz.-Mat. Ser. **2** (1956), 17–38. (Ukrainian)

The present paper is mainly concerned with the theory of optical interferometers. The analysis is not rigorous, but some of the theoretical results are in agreement with observational data.

A more rigorous and complete analysis is presented in the author's thesis [Kiev, 1949]. *K. Bhagwandin.*

Rytov, S. M. Correlation theory for Rayleigh scattering of light. I. Soviet Physics JETP **6** (1958), 401–408.

The author considers the scattering of light by fluctu-

ations in a medium. The fluctuations are considered to be both space and time dependent. Assuming the fluctuations are small, the scattered intensity for a given change in frequency, Ω, and in direction, $\boldsymbol{q}$, can be shown to equal the square of the magnitude of the four-dimensional Fourier transform of the fluctuations in the dielectric constant evaluated at Ω and $\boldsymbol{q}$. This spectral resolution of the dielectric constant in terms of the phenomenological constants of the material is taken from another paper by the author [Soviet Physics JETP **6** (1958), 130–140; MR **20**, 476] The resultant scattered intensity is evaluated for a number of specific situations.

H. Feshbach.

Rytov, S. M. Correlation theory for Rayleigh scattering of light. II. Soviet Physics JETP **6** (1958), 513–523.

In an earlier paper [reviewed above], the author has developed a theory of scattering of light in a condensed medium, based on the correlation function of the electric polarization of the medium. This function measures the correlation in the fluctuations of the polarization at neighboring points in space, and can be related to various physical properties of the substance. The present paper continues the study of the theory, with special reference to the intensities and polarizations in the scattered radiation. *E. L. Hill* (Minneapolis, Minn.).

★**Fok, V. A. Theory of diffraction from a paraboloid of revolution.** Translated by Morris D. Friedman, Inc., 67 Reservoir Street, Needham Heights 94, Mass., 1957. 41 pp.

Translated from the article in Sbornik (Collection of papers), Diffraktsia, Sovetskoe Radio, Moscow, 1957, pp. 5–56.

Heins, Albert E.; and Silver, Samuel. Comments on the treatment of the diffraction of plane waves: Addendum to "The edge conditions and field representation theorems in the theory of electromagnetic diffraction". Proc. Cambridge Philos. Soc. **54** (1958), 131–133.

Addendum to same Proc. **51** (1955), 149–161 [MR **16**, 884].

Vacca, Maria Teresa. Sulla propagazione di onde elettromagnetiche in un tubo indefinito a sezione rettangolare con dielettrico variabile periodicamente secondo l'asse. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **92** (1957–58), 73–89.

The author demonstrates the existence of propagating E-modes and H-modes in a rectangular waveguide filled with a dielectric that is a periodic function of distance along the guide. *C. H. Papas* (Pasadena, Calif.).

Bresler, A. D.; Joshi, G. H.; and Marcuvitz, N. Orthogonality properties for modes in passive and active uniform wave guides. J. Appl. Phys. **29** (1958), 794–799.

By using the formalism of linear operators, the authors deduce orthogonality relations for the guided modes of passive, anisotropic, uniform waveguides and of uniform waveguides containing longitudinal electron beams with infinite magnetic focusing fields. *C. H. Papas.*

Aymerich, Giuseppe. Sulla teoria della propagazione in una guida contenente un dielettrico giromagnetico. Rend. Sem. Fac. Sci. Univ. Cagliari **26** (1956), 165–180.

Si traduce il problem a della propagazione in una guida di sezione qualsiasi, riempita in tutto od in parte da un dielettrico giromagnetico, in un'equazione integrale di Fredholm. Si indica un procedimento per lo studio degli autovalori e si considera il caso particolare della sezione circolare. *Riassunto dell'autore.*

Svešnikov, A. G. Approximate method of computation for a weakly irregular wave-guide. Dokl. Akad. Nauk SSSR (N.S.) **110** (1956), 197–199. (Russian)

See also: Statistical Thermodynamics and Mechanics: Kadomtsev. **Structure of Matter:** Slater. **Fluid Mechanics, Acoustics:** Kahan; Jones and Whitham; Piddington; Agostinelli; Shafranov. **Quantum Mechanics:** Gor'kov. **Relativity:** Pham Mau Quan. **Control Systems:** Caldwell.

Classical Thermodynamics, Heat Transfer

v. Krzywoblocki, M. Z. On some aspects of diabatic flow and general interpretation of the wave mechanics fundamental equation. Acta Phys. Austriaca **12** (1958) 60–69.

Extending Menger's interpretation of the first law of thermodynamics [Amer. J. Phys. **18** (1950), 89–103], the author shows the existence of the total differential of a heat input function in the case of diabatic flow, which is defined as a flow of a perfect, inviscid, non-heat conducting fluid, subject to heat injection by means of heat sources distributed in the fluid medium, Also, with proper correspondances of parallel terms, Madelung's hydrodynamic interpretation [Z. Physik. **40** (1926), 322–326] of the Schrödinger equation can be extended to a diabatic flow, provided that the heat input function is of such a form that the flow is irrotational and the velocity potential exists. *L. N. Tao* (Chicago, Ill.).

Jäckel, H. Mathematische Behandlung gesteuerter Abkühl- und Anwärmvorgänge. Ing.-Arch. **26** (1958), 146–156.

Let $T(P, t)$ denote the temperature at time t at a point P inside a solid body. If the boundary of the solid consists of r surfaces F_k ($k=1, 2, \cdots, r$), a general boundary value problem in T can be written $T_t(p, t)=a^2\nabla^2T(P, t)$, $T(P, 0)=f(P)$, $\beta_kT_n(P_k, t)+T(P_k, t)=\phi_k(P_k, t)$, where p_k denotes points on F_k, T_n denotes the normal derivative, and the constants a and β_k and the functions f and ϕ_k are prescribed. The author shows how the solution of this problem can be resolved into that of solving simpler problems of two types, both suitable for using eigenvalues. One type consists of the above problem with ϕ_k replaced by zero. The other is the steady-state type $\nabla^2T=0$, $\beta_kT_n(P_k, t')+T(P_k, t')=-\phi_k(P_k, t')$, where t' is a fixed parameter and all but one of the ϕ_k's are zero. Two examples are given. [Reviewer's note: Essentially the same resolution was found by another method by R. C. F. Bartels and the reviewer, Bull. Amer. Math. Soc. **48** (1942), 276–282; MR 3, 243]. *R. V. Churchill.*

Reid, Walter P. Heat flow in a cylinder. Quart. Appl. Math. **16** (1958), 147–153.

The boundary value problem $u_t=kr^{-1}(ru_r)_r$ ($0\leq r<L$), $u(r, 0)=f(r)$, $u_r(L, t)=a[w(t)-u(L, t)]$, $kw'(t)=-u_r(L, t)+b[g(t)-w(t)]$, $w(0)=W$, is solved for $u(r, t)$ and $w(t)$. The eigenvalue problem arising here contains the parameter explicitly as a coefficient in the boundary condition,

resulting in a modified form of orthogonality of eigenfunctions. The function $u(r, t)$ represents temperatures in an infinitely long wire whose cross section has radius L. Linear heat transfer takes place from the surface $r=L$ of the wire to a thin-walled surrounding tube whose temperature is denoted by $w(t)$, and from the tube to its surroundings at temperature $g(t)$. *R. V. Churchill.*

Ruoff, Arthur L. An alternate solution of Stefan's problem. Quart. Appl. Math. **16** (1958), 197–201.

The independent variables x and t in two well-known cases of Stefan's moving boundary problem are replaced by a single variable $y=kx/\sqrt{t}$. In those special cases this substitution leads to a simple solvable problem in ordinary differential equations. *R. V. Churchill.*

Hunziker, Raul R. Heat transfer in turbulent pipe flow J. Franklin Inst. **265** (1958), 205–225.

The problem of velocity and temperature distributions in the entrance regions of turbulent flow in circular tubes and between parallel plates is investigated. The velocity distribution functions developed by Pai [J. Appl. Mech. **20** (1953), 109–119] are used to establish a partial differential equation for the temperature in terms of the radial or later and axial or longitudinal positions. The solution is reduced to a Sturm-Liouville problem by application of Laplace transforms. The corresponding eigenvalues and eigenfunctions are approximated by the Ritz method, from which the temperature and velocity distributions are obtained. Further applications are made to the determination of the local Nusselt number. *N. A. Hall.*

Dennis, S. C. R.; and Poots, G. A solution of the heat transfer equation for laminar flow between parallel plates. Quart. Appl. Math. **14** (1956), 231–236.

Cikunov, V. A. On a method of determining the coefficient of turbulent heat transfer in the upper layer of the sea. Trudy Gos. Okeanograf. Inst. **33(45)** (1956), 80–91. (Russian)

Regirer, S. A. The diffusion of the vortical layer and the exchange of heat. Dokl. Akad. Nauk SSSR (N.S.). **114** (1957), 737–740. (Russian)

Sparrow, E. M.; and Gregg, J. L. Nonsteady surface temperature effects on forced convection heat transfer. J. Aero. Sci. **24** (1957), 776–777.

Žuhovickiĭ, E. M. On the stability of a non-uniformly heated fluid in a spherical cavity. Prikl. Mat. Meh. **21** (1957), 689–693. (Russian)

Cheng, S. I.; and Kovitz, A. A. Mixing and chemical reaction in the laminar wake of a flat plate. J. Fluid Mech. **4** (1958), 64–80.

The problem of mixing and chemical reaction in the wake of a flat plate is considered to include the viscous effect by means of the boundary layer approximations. It is assumed that a cool combustible mixture and its hot combustion products are separated by a finite, perfectly insulating flat plate. At the trailing edge of the partition the velocity distribution is of the Blasius type. It is further assumed that the combustible and combustion product are of same molecular weights and all transport properties are constant.

The Howarth transformation [Proc. Roy. Soc. London Ser. A. **194** (1948), 16–42; MR **10**, 270] is used to uncouple the continuity and momentum equations from the energy and combustion equations. The method of solving these equations is parallel to that of a mixing problem (without combustion) in the wake of a flat plate with symmetric Blasius velocity profiles at the trailing edge, solved by S. Goldstein [Proc. Cambridge Philos. Soc. **26** (1930), 1–30]. Using Goldstein's velocity solutions, the temperature distribution and relative mass concentration of combustion are obtained for both near and far away from the plate in the form of infinite series. In contrast to Goldstein's perfect match, the matching of these two series for large and small distances from the plate is done with success only for the first two terms.

The paper concludes that the axial distance from the trailing edge to the first local temperature maximum is to be greatly shortened by the presence of the viscous wake as compared with non-viscous mixing. *L. N. Tao.*

Barenblatt, G. I.; and Zel'dovič, Ya. B. On stability of flame-propagation. Prikl. Mat. Meh. **21** (1957), 856–859. (Russian)

See also: Elasticity, Plasticity: Nowacki. **Fluid Mechanics, Acoustics:** Niuman and Pohlhausen; Foote; Regirer.

Quantum Mechanics

★**Séminaire de théories physiques dirigé par Louis de Broglie, 25e année: 1955/56.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. ii+145 pp. (polycopiées)

★**Landau, L. D.; and Lifshitz, E. M. Quantum mechanics: non-relativistic theory. Course of Theoretical Physics, Vol. 3.** Translated from the Russian by J. B. Sykes and J. S. Bell. Addison-Wesley Series in Advanced Physics. Pergamon Press Ltd., London-Paris; for U.S.A. and Canada: Addison-Wesley Publishing Co., Inc., Reading, Mass.; 1958. xii+515 pp. $12.50.

Translation of the Russian edition of 1948, reviewed in MR **10**, 582. Some small additions have been made and a few errors corrected.

★**Heber, Gerhard; und Weber, Gerhard. Grundlagen der modernen Quantenphysik. Teil I: Quantenmechanik.** Mathematisch-Naturwissenschaftliche Bibliothek, Bd. 1. B. G. Teubner Verlagsgesellschaft, Leipzig, 1956. vi+146 pp. DM 7.50.

★**Heber, Gerhard; und Weber, Gerhard. Grundlagen der modernen Quantenphysik. Teil 2: Quantenfeldtheorie.** Mathematisch-Naturwissenschaftliche Bibliothek, Bd. 2. B. G. Teubner Verlagsgesellschaft, Leipzig, 1957. vi+138 pp. DM 7.50.

These two thin volumes (totaling some 280 small pages) are set out to accomplish a very ambitious task: a survey of quantum theory from Planck to the latest in quantum field theory. The first volume is concerned with classical quantum mechanics including the quantum theoretical results of the first quarter of this century, while the second volume is devoted to quantization and field theory in general. The work is not a textbook. It would be impossible to write a useful textbook on this subject in such

a short space. Even less is it a reference book, although a few special topics and calculations are covered in sufficient detail to be useful as a reference. On the other hand, it is not a semi-popular account of the topic either. It is rigorous and sufficiently mathematical to be understandable only to somebody with a fair amount of background in mathematics and classical physics. Thus the question arises as to whom the book was written for, and whether it is in fact a useful kind of book at all. The answer to the latter question is an emphatic "yes". As to the former question, this reviewer sees at least two classes of people who would greatly benefit from this book. One is those students of physics who have not yet had their formal courses in quantum mechanics, and who want to get a clear and concise over-all picture of what they are about to get into. The other group might consist of those who have just gone through a series of detailed courses on various aspects of quantum theory, and who cannot quite see the forest for the trees. The book, partly because of its limited size, partly because of its clarity and conciseness, gives a fine perspective of the field as a whole. Quite a few illustrative examples are included to relieve the compactness of the book. The first volume consists of four sections. The first discusses the dualism of light, the second the dualism of matter, while the third deals with the unification of the two aspects of this dualism for a fixed number of particles (that is, in classical quantum mechanics). The fourth section is devoted to the spin of the electron, and gives a survey of the wave equations in magnetic field, the Dirac equation, and pair theory. The second volume consists of three sections. The first deals with the quantization of fields, and discusses in particular the quantization of the Schroedinger field, Dirac field, and electromagnetic field. The second section is about interactions, and gives an account of the S matrix methods, Feynman graphs, with examples like pair production and Compton scattering. The third section deals with the latest developments in quantum electrodynamics, such as self energy calculations, vacuum polarization, renormalization, and regularization. Special attention is paid to the Lamb shift and to the anomalous magnetic moment of the electron, which are quoted as experimental verifications of the correctness of the theory. Essentially nothing is said about meson field theory, and since the book was written in 1956, nothing at all is said about dispersion relations. For details the reader is referred to other books. The list of these references, however, is quite incomplete, and is, at best, geared to what a physicist in East Germany has easy access to. The second volume also has an appendix containing a useful summary of delta functions and of Green's functions for certain problems.

M. J. Moravcsik (Livermore, Calif.).

Stratonovich, R. L. On the statistical interpretation of quantum theory. Soviet Physics. JETP **5** (1957), 1206–1216.

To determine the joint probability distribution of a number of non-commuting quantities A_i, it is necessary to define uniquely in what order the operators in a product should occur. Several different ordering principles (OP) are considered, including Feynman's [Phys. Rev. (2) **84** (1951), 108–128; MR **13**, 410] chronological one (C). For the ground state, an expression is obtained for the distribution of trajectories in functional space. It is shown that the ordering principle defined by the equation

$$OP\{\exp(\sum A_i u_i)\}=C^*\{\exp \tfrac{1}{2}\sum A_i u_i\}C\{\exp \tfrac{1}{2}\sum A_i u_i\},$$

where C^* stands for the antichronological ordering, and where the u_i are 'time' parameters, corresponds to a Markov process, if it describes a time evolution. The relation between the statistical interpretation developed in this paper and relativistic field theory is discussed.

D. ter Haar (Oxford).

Schremp, E. J. Parity nonconservation and the group-space of the proper Lorentz group. Phys. Rev. (2) **108** (1957), 1076–1077.

The author proposes two modifications of his version of the Dirac equation for a spinor of spin $\frac{1}{2}$ and iso-spin $\frac{1}{2}$ [Phys. Rev. (2) **99** (1955), 1603; MR **17**, 114]. These are (A) that the 2×2 matrix representation of the quaternion units e_0, e_1, e_2, e_3 be replaced by a 4×4, (B) that the differential operators $\square_{\mp}=e_0(\partial/\partial x^0)\mp i(\sum_{j=1}^{3} e_j\partial/\partial x^j)$ be replaced by $e^{\mp 2i\mu}\square_{\mp}$, where μ is a real parameter. The resulting equations were also discussed by K. Nishijima [Nuovo Cimento (10) **5** (1957), 1349–1354]. This author relates his modifications of the equations to certain symmetry transformations introduced by Salam [ibid. **5** (1957), 299–301] to explain the zero mass of the neutrino and the non-conservation of parity. The physical interpretation is discussed somewhat cryptically.

A. S. Wightman (Princeton, N.J.).

Adams, E. N. Definition of energy bands in the presence of an external force field. Phys. Rev. (2) **107** (1957), 698–701.

Bekannterweise beruht die ganze Theorie der Leitfähigkeit von festen Körpern auf der Bändertheorie, nach der in Festkörpern verbotene und teilweise leere oder ganz volle erlaubte Energiebänder vorhanden sind. In solch einem teilweise leeren Band bewegt sich dann ein Elektron unter der Wirkung eines äusseren Feldes fast ganz so, wie wenn es frei wäre, nur muss eine scheinbare Masse eingeführt werden. In der vorliegenden Arbeit wird die Frage untersucht, wie dieses Bändersystem infolge des äusseren Feldes verändert wird. Erstens ist es klar, dass in diesem Falle die Energiebänder nicht mehr wagrecht sondern schief verlaufen werden. Zur weiteren Untersuchung dieser Frage betrachtet der Verfasser das Verhalten eines Wellenpaketes in einem schwachen elektrischen Felde, ausserdem wird zur Vereinfachung ein lineares Gitter angenommen. Die fragliche Wellenfunktion ψ wird dann nach den im ungestörten Gitter auftretenden Wellenfunktionen ψ_{nk} entwickelt:

$$(1)\qquad \psi=(a/2\pi)^{\frac{1}{2}}\int dk\sum_n \varphi_n(k)\psi_{nk},$$

a bedeutet die Gitterkonstante. Die dazugehörende vom äusseren Felde $F(t)$ gestörte Schrödingergleichung wird im Kristallimpulsraum angegeben. Sie wird dann nach einer zweimaligen unitären Transformation nach der Zeit integriert und das erhaltene Resultat wieder zurücktransformiert. Für die Eigenfunktionen des neuen durch das Feld modifizierten Bändersystems folgt dann

$$(2)\qquad \psi_{nk}^{(F)}=\psi_{nk}+\sum_{n'}\psi_{n'k}[FX_{n'n}(k)/\hbar\omega_{n'n}(k)],$$

wo die $X_{n'n}$ Matrizenelemente und die $\omega_{n'n}$ Frequenzen sind. Weiter folgt, dass auch nach einer langen Zeit das Wellenpaket in einem einzigen Band bleibt, nur verschiebt sich dessen Enveloppe im Impulsraum um $\int F dt$, wie ja das auch klassisch zu erwarten ist. Vorausgesetzt wird, dass das Wellenpaket bezüglich Impuls und Energie nur eine geringe Streuung besitzt. Die erhaltenen Resultate

werden mit die von G. H. Wannier [Phys. Rev. (2) **100** (1955), 1227; **101** (1956), 1835] verglichen und zuletzt wird noch kurz das Problem des magnetischen Feldes untersucht. *T. Neugebauer* (Budapest).

Kay, I.; and Moses, H. E. The determination of the scattering potential from the spectral measure function. IV. "Pathological" scattering problems in one dimension. Nuovo Cimento (10) **5** (1957), supplemento, 230–242.

[Parts I–III: Nuovo Cimento (10) **2** (1955), 917–961; **3** (1956), 66–84, 276–304; MR **17**, 372, 740, 971.] The solution of the Gelfand-Levitan equation for the determination of a one-dimensional potential, which gives rise to a given reflection coefficient $b(k)$ (k=wave number), is obtained for certain pathological assumptions on $b(k)$. These are: (1) $b(k)$ has poles on the positive imaginary axis in the k plane but there are no corresponding bound states and (2) $b(k)$ has no poles on the positive imaginary axis but there are bound states. Specific examples are considered and worked out. For example, a class of potentials with bound states is determined for which $b(k)\equiv 0$.
H. Feshbach (Cambridge, Mass.).

Jansen, Laurens. Tensor formalism for Coulomb interactions and asymptotic properties of multipole expansions. Phys. Rev. (2) **110** (1958), 661–669.

An expression is given for the power series expansion of the electrostatic potential of two charge distributions, in terms of certain interaction tensors and multipole moments, the method having some formal advantages over conventional procedures. It is also shown that the electrostatic interaction for cylindrically symmetric charge distributions is equivalent to that of linear charge assemblies. *A. Dalgarno* (Belfast).

Hill, E. L. State spaces in scattering theory. Phys. Rev. (2) **107** (1957), 877–883.

The author's earlier work [Phys. Rev. (2) **104** (1956), 1173–1178; MR **19**, 214] on function spaces arising in the description of a free non-relativistic particle is extended to the case of Coulomb scattering. The latter problem is solved in parabolic coordinates, and the state space resulting from this solution is examined. The problems involved with such non-separable spaces and with the existence of inequivalent representations are discussed in connection with the physical meaning of the scattering theory description. *S. Deser* (Waltham, Mass.).

Schwinger, Julian. Spin, statistics, and the TCP theorem. Proc. Nat. Acad. Sci. U.S.A. **44** (1958), 223–228.

It is shown that, with the assumption of invariance under the proper, orthochronous Lorentz sub-group and the postulate of the existence of a lowest energy state (the vacuum) for any physically realizable system, the general structure of quantum field theory implies both the connection between spin and statistics of particles and the TCP theorem. Consideration of behaviour of the Lagrange function, which occurs in the differential action principle of the theory of fields, under various transformations including complex conjugation and space-time coordinate reflexion, shows that there are two equally valid forms for the Lagrange function. If one of these allows an energy spectrum extending to $+\infty$, the other allows it to extend to $-\infty$ and no vacuum state can be defined. The resulting conclusion that no vacuum state can be defined unless the types of field which occur in nature are limited gives the connection between spin and statistics and the TCP theorem. The relation of charge reflection and complex conjugation is also developed in a suggestive manner.
C. Strachan (Aberdeen).

Bocchieri, P.; und Loinger, A. Einige Bemerkungen über die Frage der verborgenen Parameter. Z. Physik **148** (1957), 308–313.

Maslov, V. P. Degeneration on passing from a discrete spectrum to a continuous one and transition from quantum mechanics to classical mechanics. Dokl. Akad. Nauk SSSR (N.S.) **114** (1957), 957–960. (Russian)

Halbwachs, Francis. Sur le mouvement de la goutte relativiste de Bohm et Vigier. C. R. Acad. Sci. Paris **245** (1957), 1298–1301.

Halbwachs, Francis. Étude de la goutte de Bohm et Vigier en relation avec le formalisme hydrodynamique de Dirac-Takabayasi. C. R. Acad. Sci. Paris **245** (1957), 1392–1394.

Andrade e Silva, João. La représentation des systèmes de corpuscules dans l'espace physique et l'interprétation causale de la mécanique ondulatoire. C. R. Acad. Sci. Paris **245** (1957), 2018–2021.

Lochak, Georges. Signification mécanique de l'invariance de jauge de première espèce dans la représentation hydrodynamique des équations de la mécanique ondulatoire. C. R. Acad. Sci. Paris **245** (1957), 2023–2025.

Guichardet, Alain. Quelques propriétés des prévisions en mécanique quantique. C. R. Acad. Sci. Paris **245** (1957), 2488–2490.

Banerji, C. C. Sommerfeld's fine structure formula from a second order equation without Thomas correction. Indian J. Phys. **30** (1956), 525–529.

Alekseev, A. I. Covariant equation for two annihilating particles. Soviet Physics. JETP **5** (1957), 696–704.

Functional derivative techniques are used to arrive at the two-body relativistic equation for a pair of annihilating particles. The interaction operator is derived to order α^2, and coincides, for the electron-positron system, with that given by Karplus and Klein [Phys. Rev. (2) **87** (1952), 848–858]. The extra contribution to this operator from the real pair annihilation possibility is also given. These results are applied to the pair annihilation (or creation) process, with emphasis on the pair's interaction in the initial (or final) state and the relevant radiative corrections. The amplitudes for these processes are given formally. *S. Deser* (Waltham, Mass.).

Norton, Richard E.; and Klein, Abraham. Complete set of dispersion relations for a class of fixed-source meson theories. Phys. Rev. (2) **109** (1958), 584–600.

Dispersion relations are constructed for a class of fixed-source meson theories, explicit use being made of the Hamiltonian's form. The interaction consists of an arbitrary finite power-series in the meson field amplitude, only S-wave production being allowed. The dispersion relations still involve the non-physical region of energy, but the complete set can be solved by iteration in terms of

arbitrary constants, the number of which equals that of the coupling constants in the Hamiltonian (the linear term, being no true interaction, is not counted). However, there exists a whole class of solutions to these coupled equations, that is, different Hamiltonians are exhibited which have the same dispersion relations; these theories differ from the original one in that they involve sources with excited states, and thus have extra resonances occurring. It is suggested that the iterative solution corresponds to the original Hamiltonian. *S. Deser.*

Omnes, R. Production en paire des mésons positifs par un méson. Nuovo Cimento (10) **6** (1957), 780–795.

A system of generalized Low equations for the static Chew-Low model is given, which includes amplitudes with arbitrary numbers of incoming and outgoing mesons. The equations for the production of a pair of positive mesons in a meson-nucleon collision are then studied in detail in the static model; they obey unitarity, causality and crossing symmetry requirements. In a one-meson approximation the resulting linear integral equation is solved numerically; its kernel is taken as a function of the experimental meson-nucleon phase shifts. For the specific charge and angular states considered, only the accurately known δ_3 is required. It is stated that an iteration solution for the equation would have diverged. The cross-section is given up to total energies of 400 MEV in the CM system, and seems in better agreement with preliminary experimental results than the smaller one obtained by previous authors by iteration approximations.

S. Deser (Waltham, Mass.).

Ivanenko, D. Non-linear generalizations of the field theory and the constant of minimal length. Nuovo Cimento (10) **6** (1957), supplemento, 349–355.

A series of remarks, given by the author at the Turin conference in 1956, concerning the possible new constants arising from non-linear self-interaction of fields and from the existence of a minimum length. Non-linear terms arising from vacuum fluctuations in usual field theory, as well as possible intrinsic ones, are discussed; Dirac equations with resultant cubic terms are considered in this light. In connection with the minimum length problem, an attempt is described to obtain the length from electromagnetic mass-difference terms in perturbation theory with the same cutoff for pions and nucleons. The resulting length is slightly greater than the nucleon Compton radius, and is proposed as the limit to the usual space-time concepts. *S. Deser* (Waltham, Mass.).

Pugh, Robert E. Furry's theorem for very strong interactions. Phys. Rev. (2) **109** (1958), 989–990.

It is shown that for the Gell-Mann-Schwinger model involving symmetric coupling of all baryons to pions, but disregarding the (weaker) k-meson interactions, there exists an analog to Furry's theorem: any closed loop with an odd number of external pion lines does not contribute, the number of photon lines being unrestricted. The proof involves the fact that there exist, in this model, particle pairs identical electromagnetically, but oppositely coupled to neutral pions; their contributions cancel each other if the splitting among the baryons is neglected.

S. Deser (Waltham, Mass.).

Namiki, Mikio; and Iso, Chikashi. Multiple production of particles and hydrodynamical aspect of quantum theory of fields. Progr. Theoret. Phys. **18** (1957), 591–613.

Landau's theory of multiple production [Izv. Akad. Nauk Ser. Fiz. SSSR **17** (1953), 51–64] regards the meson-nucleon cloud after the collision as a high temperature fluid, which expands and cools off until real particles can be formed. In the present paper, the theory of such a quantum-mechanical fluid at high temperature is developed. First the statistical mechanics of an N-particle system is formulated for relativistic velocities, and the local heat flow and internal stress tensor are defined. Next the various cases of quantized fields are treated. This leads to the conclusion that Landau's relation (pressure equals one third of energy density) is true for scalar or pseudoscalar coupling, but not for vector or pseudovector coupling. Thirdly, in order to investigate whether the assumption of local thermal equilibrium is justified, the irreversible effects are studied. The entropy production is written in terms of fluxes and forces, so that the phenomenological equations can be obtained. The phenomenological coefficients are expressed, in a well known way, as auto-correlation functions, but they are not evaluated. Finally, the authors discuss the question whether the fluctuations are indeed so small that the hydrodynamical picture is valid. *N. G. van Kampen.*

Stephen, M. J. Double refraction phenomena in quantum field theory. Proc. Cambridge Philos. Soc. **54** (1958), 81–88.

The phenomena of double-refraction, i.e., Kerr effect, optical and Faraday rotation, etc., are derived from quantum electrodynamics by treating them as a scattering problem in second order perturbation theory and by using the Stokes operator to describe polarization.

F. Rohrlich (Baltimore, Md.).

Gamba, A. Cooperative phenomena in quantum theory of radiation. Phys. Rev. (2) **110** (1958), 601–603.

Dicke [same Rev. (2) **93** (1954), 99–110] has discussed in some detail the effect of correlations between individual molecules, for example in the emission of coherent radiation by a gas of linear dimension small in comparison with the wavelength of the radiation. Here the basic applications of the group of unitary transformations in the space spanned by the state vectors (excited and non-excited) and the symmetric group for identical molecules are underlined. Important practical consequences, e.g., for a gas maser, are mentioned. *C. Strachan.*

Lomsadze, Yu. M. Effective method for calculation of an S-matrix in quantum field theory. I. Užgorod. Gos. Univ. Naučn. Zap. **18** (1957), 155–175. (Russian)

Lomsadze emphasizes difficulties which one meets when applying various kinds of series representation to functions appearing in the relativistic quantum theory of fields. He refers to such representations as MacLaurin series, perturbation techniques, etc. To overcome these difficulties, Lomsadze proposes a "modified" series, each term of which is a function of the arguments, which arguments refer to the characteristic properties of the field in question. The present paper constitutes Part I of the work on this technique. In it Lomsadze discusses the convergence of this series in an interval and its analytic continuation outside the interval in question. The technique is applied to the evaluation of S-Matrices. At the end of this part, Lomsadze discusses briefly the physical notions related to this technique. It is mentioned that Part II will contain the proof that in some cases the modified series converges absolutely. *M. Z. v. Krzywoblocki.*

Takahashi, Y.; and Umezawa, H. A general treatment of expanding systems. I. Formulation. Nuovo Cimento (10) **6** (1957), 1324–1334.

It is shown that the canonical variables in a quantised field description of a system of particles in a given volume are related by a unitary transformation to the corresponding variables for the same system in a larger volume. The unitary transformation is exhibited. *D. Falkoff.*

Takahashi, Y.; and Umezawa, H. A general treatment of expanding systems. II. Application to multiple meson processes. Nuovo Cimento (10) **6** (1957), 1382–1391.

The theory of the previous paper is applied to the problem of multiple meson production in high energy collisions. This provides a field-theoretic interpretation of the hydrodynamical theory of S. Z. Belen'kij and L. Landau [Nuovo Cimento (10) **3** (1956), supplemento, 15–31] for such processes. *D. Falkoff* (Waltham, Mass.).

Grosjean, C. C. Further development of a new approximate one-velocity theory of multiple scattering. Nuovo Cimento (10) **5** (1957), 83–101.

The solution of the transport equation can be expressed as a series in tensors of increasing rank, beginning with the scalar density, ρ, the second group of terms the current density ρv_i, the third group of terms the tensor of second rank $\rho v_i v_j$, and so on. In this paper the series is terminated at the tensor of second rank. The resulting equation for ρ can be determined, as well as expressions for the current density in terms of ρ. The details are carried out for slight anisotropic scattering and for a distribution of isotropic sources. *H. Feshbach* (Cambridge, Mass.).

Gor'kov, L. P. Two limiting momenta in scalar electrodynamics. Soviet Physics. JETP **5** (1957), 167–169.

A study is made of the possiblity of introducing two limiting momenta in scalar electrodynamics. It is shown that the difficulties connected with the vanishing of the renormalized charge in the limit of a point interaction appear also in the theory considered here.

Author's summary.

Sudakov, V. V. Vertex parts at very high energies in quantum electrodynamics. Soviet Physics. JETP **3** (1956), 65–71.

A method is developed for calculating Feynman integrals with logarithmic accuracy, working to any order of perturbation theory. The method is applied to calculate the vertex part in quantum electrodynamics for a certain range of values of the momenta. The result is displayed as the sum of a perturbation series. *Author's summary.*

Baz', A. I. Expansion of single particle wave functions in functions of the relative motion of the nucleons. Soviet Physics. JETP **4** (1957), 704–708.

For small energies of relative motion, identical nucleons interact with each other only when their relative orbital momentum is zero. Therefore, it can be expected that a nucleon in the nucleus does not interact with all of the nucleons, but only with those three which are in an S-state relative to it. These considerations lead to a picture which is in many respects similar to the α-particle model of the nucleus. The wave functions of the shell model are investigated from this point of view and a number of common properties of the shell and α-particle models of the nucleus are shown. Several regularities in the binding energies of light nuclei are discussed.

Author's summary.

Stueckelberg, E. C. G. Théorie de la radiation de photons de masse arbitrairement petite. Helv. Phys. Acta **30** (1957), 209–215.

If the photon has a mass $\hbar\mu \neq 0$, a canonical transformation yields a Hamiltonian containing the energies: (1) of the charged field (including the Yukawa potential energy); (2) of the transversal and longitudinal photons; and (3) an interaction term. The interaction term between charge and longitudinal photons is smaller than the transversal term by a factor μ/ω, if $\hbar\omega$ is the photon energy. Thus the limit $\mu \to 0$, which replaces the Yukawa potential by the Coulomb potential, is possible. The longitudinal photons, presenting no interaction, can pass freely through the walls of a cavity and Planck's radiation law holds. The necessary and sufficient condition for this result is the continuity equation for electric charge.

Author's summary.

Destouches-Aeschlimann, Florence. L'électromagnétisme non linéaire et les photons. C. R. Acad. Sci. Paris **244** (1957), 3034–3036.

Takabayasi, Takehiko. Nouvelle propriété symétrique et nouvelles relations de conservation du champ de Dirac. C. R. Acad. Sci. Paris **246** (1958), 64–66.

Shepanski, J. R.; and Butler, S. T. An expansion for Coulomb wave functions. Nuclear Phys. **1** (1956), 313–321.

The general radial Coulomb wave functions denoted by $\bar{F}_L(\rho;\eta)$ are defined as solutions of the radial Coulomb wave equation

$$(1) \qquad \left[\frac{d^2}{d\rho^2}+1-\frac{2\eta}{\alpha}-\frac{L(L+1)}{\rho^2}\right]\bar{F}_L(\rho;\eta)=0,$$

where $\rho=kr=(2\pi\cdot\text{wave number})\cdot(\text{radial distance})$; $\eta=ZZ'e^2/\hbar v$; v is the relative velocity of the two particles; and L is the orbital angular momentum quantum number.

The author studies the regular solutions of (1),

$$(2) \qquad F_L(\rho;\eta)=\frac{2^L}{(2L+1)!}\left[\prod_{m=0}^{L-1}\{(L-m)^2+\eta^2\}\frac{2\pi\eta}{e^{2\pi\eta}-1}\right]^{1/2}\rho^{L+1}\Phi_L(\rho;\eta),$$

with $\Phi_L(0;\eta)=1$, and obtains, by an iteration procedure, the expansion

$$\Phi_L(\rho;\eta)=\left\{\sum_{m=0}^{\infty}a_{L,m}(\eta)\rho^m\right\}\exp\left[\frac{\eta\rho}{L+1}-\frac{\rho^2}{4(L+1)}\right],$$

valid for all finite values of ρ. The coefficients $a_{L,m}$ are polynomials in η of order m, with rational coefficients (functions of L); the first five coefficients are computed. It is shown that the series (2) is much more rapidly convergent than the usual power series expansion

$$\Phi_L(\rho;\eta)=\sum_{k=0}^{\infty}A_{L,k}(\eta)\rho^k.$$

In general, the first one or two terms of (2) will suffice, particularly for large values of L. *S. C. van Veen.*

Jackson, J. D.; Treiman, S. B.; and Wyld, H. W., Jr. Note on relativistic Coulomb wave functions. Z. Physik **150** (1958), 640–647.

Certain solutions of the Dirac equation for a charged particle of spin one-half (electron) in a Coulomb field are considered. These are specified by the charge of the electron and by their asymptotic form, a plane wave plus an outgoing or ingoing spherical wave. They are linear combinations of the solutions given by Rose [Phys. Rev. **51**(1937), 484–485]. It is pointed out that at the origin (location of the Coulomb field) these solutions take on a very simple form, similar to the free-field solutions. The corresponding Casimir projection operators are constructed and compared with those obtained from the Furry-Sommerfield-Maue wave functions.

F. Rohrlich (Baltimore, Md.).

Gupta, Suraj N. Anomalous magnetic moments of nucleons in nucleon-photon interaction. Phys. Rev. (2) **107** (1957), 1442–1445.

The validity of a Pauli anomalous magnetic moment term as a description of the effect of the nucleon's meson cloud on nucleon-photon interaction is examined for $ps(ps)$ meson theory. This is done by calculating the energy dependence of the corresponding terms in the meson theoretic formulation. On the basis of the simplest diagrams, it is found that even for a few hundred MEV photons, the use of the static phenomenological moment may lead to large errors. *S. Deser.*

Bopp, F.; und Werner, E. Exakte Grundlagen der Theorie der Spinwellen. Z. Physik **151** (1958), 10–15.

The theorem is proved that equations of the Block type of spin wave equations arise in the exact many-electron problem when spin-orbit and spin-spin interactions are neglected. The theorem can be applied not only to solids but also to atoms and molecules. It cannot be applied directly to atomic nuclei since in this case the spin interactions are not negligible. (Translation of author's abstract). *C. Froese* (Vancouver, B.C.).

Inopin, E. V. Neutron scattering by non-spherical nuclei. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 239–241. (Ukrainian. Russian and English summaries)

This paper presents the results of a calculation of neutron scattering by non-spherical nuclei. The calculations were performed by a method which is not limited in its applicability either by the case of small deformations of the nuclei, or by the case of neutrons of high energy. The method is based on the precise solution of Schroedinger's equation in spheroidal coordinates .The calculations were found to be substantially simplified in the case of considerable absorption in the nucleus. *Author's summary.*

Moshinsky, Marcos. Nuclear forces with hard core and the shell model of the nucleus. Second order effects. Rev. Mexicana Fis. **6** (1957), 185–199. (Spanish. English summary)

Metropolis, N.; Bivins, R.; Storm, M.; Turkevich, Anthony; Miller, J. M.; and Friedlander, G. Monte Carlo calculations on intranuclear cascades. I. Low-energy studies. Phys. Rev. (2) **110** (1958), 185–203.

Metropolis, N.; Bivins, R.; Storm, M.; Miller, J. M.; Friedlander, G.; and Turkevich, Anthony. Monte Carlo calculations on intranuclear cascades. II. High-energy studies and pion processes. Phys. Rev. (2) **110** (1958), 204–219.

Watanabe, Shigueo. On a determination of the characteristic parameters of a level of nuclear resonance. An. Acad. Brasil. Ci. **28** (1956), 243–251. (Portuguese)

Schwinger, Julian. Dynamical theory of K mesons. Phys. Rev. (2) **104** (1956), 1164–1172.

Accepting Gell-Mann and Nishijima's assignment [M. Gell-Mann and A. Pais, Proceed. 1954 Glasgow Conference on Nuclear and Meson Physics, Pergamon Press, London-New York, 1955, pp. 342–352; K. Nishijima, Progr. Theoret. Phys. **12** (1954), 107–108] of $T=\frac{1}{2}$ for $N=(p, n)$, (Ξ^0, Ξ^-), and (K^+, K^0) isotopic doublets, $T=0$ for Λ and $T=1$ for $(\Sigma^+, \Sigma^0, \Sigma^-)$ multiplet, the following suggestions are made —: (1) All K-hyperon couplings have the identical strength. This implies a four-dimensional internal symmetry for these interactions. (2) The π-meson is coupled directly to (p, n) and (K^+, K^0) particles but not to (Ξ^0, Ξ^-), thus upsetting the four-dimensional symmetry of the inner (isotopic) space. If this interaction were to be removed, N and Ξ would become a degenerate multiplet. Further, since π-meson is pseudoscalar, a direct three-field interaction with K-mesons is possible only if both scalar (θ) and pseudoscalar (τ) K-mesons exist. (3) With the plausible assumption that $\theta \to \mu$, ν and $\tau \to \mu$, ν decay couplings are identical, and the assumption that $K \to \pi + e + \nu$ decay mode is to be ascribed entirely to τ-decay, the ratio of τ to θ life-time is calculated as 4/3 from the known abundances of the various K-decay modes ($\tau \to 3\pi$, $\theta \to 2\pi$, $K \to \mu, \nu$ etc.). The reviewer has learnt that in his recent version of the theory the author has modified these proposals in view of recent experiments [High Energy Nuclear Physics, Proc. 7th Annual Rochester Confer., 1957, Interscience, New York, 1957, pp. IX–15–IX–22]. *A. Salam* (London).

Feshbach, Herman; and Villars, Felix. Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles. Rev. Mod. Phys. **30** (1958), 24–45.

In a frankly pedagogical article, the authors describe the single particle relativistic quantum mechanics for spin 0 and spin $\frac{1}{2}$, and the connection with quantum field theory. As is well known, a single particle theory is only possible when the interaction with the outside world is sufficiently weak for it to be possible to neglect the effects of other particles, real or virtual. In effect, this implies a restriction to certain external electromagnetic fields.

The Klein-Gordon Equation is treated as a two component system (one component for each charge state), and free particle solutions, position operators (Zitterbewegungs) and effects of weak electromagnetic fields are discussed. The corresponding and more familiar results of the Dirac Equation are also presented. *J. C. Ward.*

Rayski, Jerzy. On a screw model of particles in isospace. Acta Phys. Polon. **16** (1957), 279–291. (Russian summary)

The charge, isotopic spin, and strangeness quantum numbers of strongly interacting particles are defined by use of an abstract isotopic spin space. In terms of a rule differing somewhat from the usual one relating the above quantum numbers, a classification is made of the particles.

The space is given more structure than in other models, as particles are taken to possess a spinning center and outer point, orbital angular momentum and a size in it. The rest masses are also related to these internal parameters by means of field equations containing eigenvalue equations for the masses. It is attempted to include leptons in this scheme, but since isotopic spin is then not conserved, a new particle, the "isotrino", is defined; it possesses only isotopic quantum numbers and accounts for this apparent non-conservation by always being emitted in leptonic processes. *S. Deser.*

Polkinghorne, J. C. On the strong interactions. Nuovo Cimento (10) **6** (1957), 864–868.

The four-dimensional isotopic spin formalism originally proposed by Salam and Polkinghorne [Nuovo Cimento (10) **2** (1955), 685–690; MR **19**, 1018] is taken up again on the basis of the Gell-Mann and Schwinger suggestions that the very strong (VS) baryon-pion interactions be invariant under rotation in a four-dimensional space, and the medium-strong (MS) k-meson-baryon coupling be invariant in a three-dimensional subspace only. The author's scheme differs from those of Schwinger and Gell-Mann in that different groupings of the particles as representations of the rotation group are made. The VS interaction is written down and involves three independent coupling constants. For the MS case, the interaction written down is an arbitrary three-dimensional one. This is necessary in order to lift the degeneracy between nucleon and cascade particle with the representations used. *S. Deser* (Waltham, Mass.).

Borgardt, A. A. Nonlinear meson field equations. Soviet Physics. JETP **6** (1958), 43–49.

The effects of third-order self-interaction terms in pseudoscalar meson field theories are investigated. These include, besides the familiar φ^3 term, self-couplings involving both field amplitudes and their gradients in various combinations. All such terms possible are obtained from the Kemmer form and by fusion of two spin $\frac{1}{2}$ fields. Static solutions and their singularities are examined. The domains of coupling constant for which the wave equations admit of periodic solution are also noted.
S. Deser (Waltham, Mass.).

Freistadt, H. The causal formulation of quantum mechanics of particles (the theory of De Broglie, Bohm and Takabayasi). Nuovo Cimento (10) **5** (1957), supplemento, 1–70.
Expository article. *C. A. Hurst* (Adelaide).

Fradkin, E. E. On the Rarita-Schwinger method in the theory of particles of half-integral spin. Soviet Physics. JETP **5** (1957), 1203–1205.

It is shown that if we try to derive a Rarita-Schwinger type of field equation for particles of half-integral spin from a Lagrangian density by the variational principle, serious difficulties appear in the case of particles of spin greater than 3/2. *S. N. Gupta* (Detroit, Mich.).

Goldberger, M. L.; and Treiman, S. B. Decay of the pi meson. Phys. Rev. (2) **110** (1958), 1178–1184.

The decay rate for the process $\pi \to \mu + \nu$ is calculated by assuming that the pion first disintegrates into a virtual nucleon-antinucleon pair, which then produces the lepton pair. The interaction of the pion and the virtual nucleon-antinucleon pair is treated by using the techniques of the dispersion theory. After making several plausible assumptions throughout the treatment, a theoretical value for the pion lifetime is obtained, which is in close agreement with the experimental result. *S. N. Gupta.*

Baker, M. Determinantal approach to meson-nucleon scattering. Ann. Physics **4** (1958), 271–305.

A determinantal method for the solution of scattering problems is described, which is essentially equivalent to the Fredholm solution of the scattering integral equation. As an example, the meson-nucleon scattering in the static model is calculated with the one-meson approximation, and the results are found to be in agreement with the earlier work of Chew and Low [Phys. Rev. (2) **101** (1956), 1570–1579]. *S. N. Gupta* (Detroit, Mich.).

Bollini, Carlos G. Intensity and selection rules for multipolar radiation. Repub. Argentina. Publ. Com. Nac. Energia Atomica. Ser. Fis. **1** (1956), 129–166. (Spanish. English summary)

Interaction between an atomic electron and a radiation field is treated by representing the latter by means of electrical and magnetic spherical solutions from Maxwell equations. In this way a suitable formalism for the attainment of intensities and the corresponding selection rules is achieved, taking into account the conservation of total angular momentum. *From the author's summary.*

Mohan, G. On the creation operators of physical particles. Nuovo Cimento (10) **7** (1958), 491–500.

The conventional annihilation and creation operators of free particles are modified in view of the fact that all measurements require non-vanishing duration of time. The modified operators create and annihilate physical particles. Asymptotic assumption or adiabatic hypothesis are not invoked so that the renormalized S-matrix can be obtained without their help. The analysis of certain functions that occur in the definitions of the modified operators is incomplete. *Author's summary.*

Pluvinage, Philippe; et Proriol, Joseph. Sur la relation portée-profondeur dans le problème du deutéron. C. R. Acad. Sci. Paris **245** (1957), 2201–2203.

Aeschlimann, Florence. Extension de la théorie fonctionnelle des corpuscules au nucléon et au photon. J. Phys. Radium (8) **18** (1957), 562–566.

Prieto, Fernando E. Solutions of Bhabha's equation for a free particle. Rev. Mexicana Fis. **6** (1957), 165–184. (Spanish. English summary)

See also: Numerical Methods: Dettmar and Schlüter. **Statistical Thermodynamics and Mechanics:** Lax; Klimontovich and Temko. **Optics, Electromagnetic Theory, Circuits:** Schlüter. **Relativity:** Misner.

Relativity

Schöpf, Hans-Georg. Die Energiestromgeschwindigkeit im elektromagnetischen Feld. Z. Physik **148** (1957), 417–424.

Etant donné un tenseur d'impulsion-énergie arbitraire, l'A. étudie, conformément à une suggestion de v. Laue [Z. Physik **128** (1950), 387–394; MR **12**, 377] reprise par Møller [The theory of relativity, Oxford, 1952, § 62; MR

14, 212], à quelles conditions le quotient de la densité de courant d'énergie par la densité d'énergie satisfait au théorème d'addition des vitesses d'Einstein. Il indique un système simple de conditions nécessaires et suffisantes par qu'il en soit ainsi et applique ces conditions au tenseur électromagnétique classique de Minkowski, en présence d'induction. La propriété étudiée est satisfaite pour les champs purement radiationnels. Un autre tenseur électromagnétique proposé indépendammente par Marx et Györgyi [Acta Phys. Acad. Sci. Hungar. 3 (1954), 213–242; MR 16, 775] et par Pham-Mau-Quan est analysé.

A. Lichnerowicz (Paris).

Palacios, Julio. Should the theory of relativity be revised? An. Real Soc. Españ. Fis. Quim. Ser. A. **53** (1957), 31–42. (Spanish)

Pham Mau Quan. Sur les équations de l'induction électromagnétique. C. R. Acad. Sci. Paris **246** (1958), 707–710.

Unal, Burhan Cahit; et Vigier, Jean-Pierre. Introduction des paramètres relativistes d'Einstein-Kramers et de Cayley-Klein dans la théorie relativiste des fluides dotés de moment cinétique interne (spin). C. R. Acad. Sci. Paris **245** (1957), 1785–1788.

Andreoli, Giulio. Composizione relativistica di moti effettivi di particelle e di moti associati di de Broglie. Ricerca, Napoli (2) **8** (1957), Luglio-Dicembre 3–7.

Bel, Louis. Sur les discontinuités des dérivées secondes des potentiels de gravitation. C. R. Acad. Sci. Paris **245** (1957), 2482–2485.

Lichnerowicz, André. Sur les ondes et radiations gravitationnelles. C. R. Acad. Sci. Paris **246** (1958), 893–896.

Trautman, André. Sur la propagation des discontinuités du tenseur de Riemann. C. R. Acad. Sci. Paris **246**(1958), 1500–1502.

A Riemannian space-time satisfying the differentiability and continuity conditions proposed by Lichnerowicz [Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955, Ch. 1; MR **17**, 199] is called a Lichnerowicz space-time. Lichnerowicz showed that in such a space-time the characteristics of Einstein's field equations are null hypersurfaces (tangent to null cones $ds^2=0$) and that the bicharacteristics are null geodesics. The reviewer [Phys. Rev. (2) **105** (1957), 1089–1099] calculated the discontinuities $[R_{\mu\nu\rho\sigma}]$ in the Riemann tensor across a null hypersurface $\Sigma:\varphi=$ constant, and interpreted such hypersurfaces physically as gravitational wave-fronts. Trautman [Bull. Acad. Polon. Sci. Cl. III. **5** (1957), 273–277, XXIII; MR 19, 509] generalized these results as follows: If $[\cdot]$ denotes discontinuity, then for any (not necessarily null) hypersurface Σ Lichnerowicz's conditions require $[g_{\alpha\beta,\gamma\delta}]=h_{\alpha\beta}\varphi_{,\gamma}\varphi_{,\delta}$ where $h_{\alpha\beta}$ is a tensor on Σ. Hence $[R_{\mu\nu\rho\sigma}]=-\frac{1}{2}\delta_{\mu\nu}{}^{\alpha\beta}\delta_{\rho\sigma}{}^{\gamma\delta}h_{\alpha\gamma}\varphi_{,\beta}\varphi_{,\delta}$. If the matter tensor is continuous across Σ: $[R^{\mu}{}_{\nu\rho\mu}]=0$, then $[R_{\mu\nu\rho\sigma}]$ vanishes unless Σ is null. If Σ is null then there are 4 linear relations between the 10 components of $h_{\alpha\beta}$.

In the first of the papers under review, these linear relations are put into the form $(h_\alpha{}^\beta-\delta_\alpha{}^\beta h_\gamma{}^\gamma)\varphi_{,\beta}=0$ (where $h_\alpha{}^\beta=g^{\beta\gamma}h_{\alpha\gamma}$). It is deduced that $h_{\alpha\beta}=s(v_\alpha v_\beta-w_\alpha w_\beta)$, where s is scalar and v_α and w_α are spacelike vectors orthogonal to $\varphi_{,\alpha}$ and to each other (a covariant version of a result of the reviewer). An extension to the Jordan-Thiry theory is indicated.

In the second of the papers under review, it is shown that $[R_{\mu\nu[\rho\sigma}]\varphi_{,\tau]}=0$ and that $[R^{\mu}{}_{\nu\rho\sigma}]\varphi_{,\mu}=0$. The theory is extended to include Maxwell fields.

In the third of the papers under review, it is shown that along a bicharacteristic $dx^\alpha/du=g^{\alpha\beta}\varphi_{,\beta}$, $[R_{\mu\nu\rho\sigma}]$ is propagated according to the equation

$$2\frac{\delta}{\delta u}[R_{\mu\nu\rho\sigma}]+[R_{\mu\nu\rho\sigma}]\,\Box\varphi=0$$

(where $\delta/\delta u$ is the absolute derivative and $\Box$ the covariant wave operator). As an example, a discontinuity in Schwarzschild space-time is considered.

F. A. E. Pirani (Baltimore, Md.).

Kronsbein, John. Relativity in static spherical and elliptic space (Einstein's universe). Phys. Rev. (2) **109** (1958), 1815–1822.

The Einstein model of the universe is treated by a new method, which depends on the introduction of coordinates which differ from those which had been used in the past by Einstein and others. Let S_R be the hypersphere of radius R in a euclidean E_4 with Cartesian coordinates x_i ($i=1, \cdots, 4$; $\alpha=1, 2, 3$). Two parametric representations of S_R are considered:

(1) $x_\alpha=RX^\alpha\Delta^{-1}$, $x_4=R^2\Delta^{-1}$, where $\Delta=R^2+\Sigma(X^\alpha)^2$

(radial projection of S_R from E_4 into the elliptic space ε_3 with contravariant coordinates X^α), $X^4=ct$;

(2) $x_1=2R^2x/(R^2+r^2)$, $x_2=2R^2y/(R^2+r^2)$,

$x_3=2R^2z/(R^2+r^2)$, $x_4=R(R^2-r^2)/(R^2+r^2)$,

where $r^2=x^2+y^2+z^2$ (stereographic projection of S_R upon $E_3{}^{(a)}$ with Cartesian coordinates (x, y, z), with the proviso that this space be closed at infinity by a single point). It is shown that the group of transformations on static spherical or elliptic space-time X^4 or, also, the group on the Einstein universe, has a subgroup which is simply isomorphic to the special Lorentz group; furthermore, a more extensive subgroup exists which is simply isomorphic to the full (proper and future-preserving) Lorentz group. — The author's method accentuates the structure of the Einstein model in the large, and he obtains significant geometrical results, a description of which is not feasible within the limits of a review.

H. Rund (Durban).

Misner, Charles W.; and Wheeler, John A. Classical physics as geometry: Gravitation, electromagnetism, unquantized charge, and mass as properties of curved empty space. Ann. Physics **2** (1957), 525–603.

The programme of reducing classical electromagnetism, together with gravitation, to pure Riemannian geometry is advanced by (1) re-constructing an electromagnetic vacuum field from its stress tensor, and hence from the Ricci curvature tensor; (2) identifying charges, i.e. apparent sources and sinks of electric flux, as junctions of topological "handles" to our (nearly) flat space: unending flux lines converge upon one junction, pass through the handle and emerge from the other.

The electromagnetic field is re-constructed from its stress tensor by taking the antisymmetric "Maxwell square root": reduce $T_{\mu\nu}$ locally to principal axes (the

principal stresses are equal in magnitude, opposite in pairs, so that the trace vanishes) and place $E=(8\pi T_{00})^{\frac{1}{2}}$ along the space axis associated with stress $+T_{00}$. Transform back to the original observer's frame, introducing an H perpendicular to E.

The field so obtained may be subjected to a "duality rotation"

$$E\to E\cos\alpha+H\sin\alpha,\quad H\to -E\sin\alpha+H\cos\alpha,$$

which does not change the stresses. The "Maxwell square root" is arbitrary up to the angle α, termed the "complexion" of a field ($-\alpha$ determines that duality rotation which makes E and H mutually perpendicular; it is invariant). However, Maxwell's equations lay down a definite rate of change for α in space-time so that there is arbitrariness only up to a universal complexion constant.

The existence of scalar complexion $\alpha(x^\nu)$ obeying Maxwell's equation entails a condition also on the derivatives $T_{\mu\nu;\lambda}$ and hence on the derivatives of the Ricci tensor components. A certain vector formed from these must be a gradient. Ricci tensors which have characteristic roots equal in magnitude and obey this condition contain, implicitly, a vacuum electromagnetic field. This constitutes the authors' "already unified field theory".

The theory is, further, formulated in a non-singular manner in spaces of arbitrary topology: a self-sufficient summary of the necessary topological concepts and theorems is given. Vectors, cross-products, differentiation and integration are defined and Stokes' theorem is formulated. Electric and magnetic flux through "closed" surfaces is defined. The flux around the junction of a "handle" (termed the "mouth" of a "wormhole") is shown to be a constant attribute of the handle or wormhole, its charge or pole strength. By the theorems of de Rham the existence of such flux carrying wormholes in 4-space is proved, as well as the existence of a vector potential, when none of the holes trap magnetic flux. (For one hole, i.e., two mouths, the absence of magnetic flux is merely a matter of adjusting the universal complexion constant, but for several holes this absence remains unexplained.)

The Cauchy initial value problem is discussed and a generalised Fourier analysis is used for calculating the evolution of the electromagnetic field in curved, multiply connected space. The Reissner-Nordstrom solution for the exterior of a charge-mass-centre is presented, but the removal of its singularity leads to an unrealistic charge-mass ratio.

"Wormholes" carry mass, like "geons" [J. A. Wheeler, Phys. Rev. (2) **97** (1955), 511–536; MR **16**, 756], but the latter exist in singly connected space. None of these entities are claimed to bear any resemblance to observed "dressed" particles. (The article is followed by a non-mathematical article by Wheeler discussing the effect of Feynmann quantisation on this field theory and the possible emergence of more realistic particle models.)

The title "Geometrodynamics" given by the authors to their theory is misleading: there is no mathematical derivation of the equations of motion of wormhole mouths. The response of a mouth to the effects on the metric of the extremely tenuous mass density $E^2/8\pi c^2$ of an external field, and hence the dynamical equations of Lorentz, remain unexplained.

The article appeals, not only through the beauty of the theory, but also through the rich and colourful presentation, through philosophical discussions and quotations and many useful references. *O. Buneman.*

Misner, Charles W. Feynman quantization of general relativity. Rev. Mod. Phys. **29** (1957), 497–509.

This paper forms part of a series by J. A. Wheeler and the author [see the preceding review]. Their point of view is that the difficulties encountered in the canonical quantization of general relativity [see Bergmann, Rev. Mod. Phys. **29** (1957), 352–354] may be avoided by writing the transition amplitude between field configurations f_1, f_2 on initial and final hypersurfaces σ_1, σ_2 as a sum over histories, according to the method of Feynman. That is, $\langle f_2\sigma_2|f_1\sigma_1\rangle=\sum_H \exp iI_H$, where I_H is the action integral for the gravitational field, evaluated along the field history H. The transition amplitude between any two initial and final states is then written as

$$\langle 2|1\rangle=\int \psi_2^*(f_2)\langle f_2\sigma_2|f_1\sigma_1\rangle\psi_1(f_1)\delta f_2\delta f_1,$$

where physical states are to be distinguished by the property that $\psi(f)$ is constant over those field configurations which differ merely by a coordinate transformation.

The author makes the sum over histories more precise by writing it as a functional integral over metrics $g_{\mu\nu}$:

$$\langle f_2\sigma_2|f_1\sigma_1\rangle=\int\delta[g_{\mu\nu}]\exp\Big\{i(c^3/16\pi G\hbar)\int d^4xR(-g)^{1/2}\Big\},$$

where R is the curvature scalar. He then discusses two problems: (i) The determination of the appropriate measure in the function space of all metrics, i.e., the meaning of $\delta[g_{\mu\nu}]$; (ii) The attainment of information on the structure of $\langle f_2\sigma_2|f_1\sigma_1\rangle$ from the topological invariance of the theory. With regard to (i), it is well known that the technique of finding the absolute measure by an appeal to the canonical formalism [see, e.g., DeWitt, same Rev. **29** (1957), 377–397] fails when applied to gauge invariant theories such as general relativity. The author therefore proposes to determine the relative measure (i.e., the dependence on $g_{\mu\nu}$) by finding the invariant measure for integration over a "homogeneity group" of the fields $g_{\mu\nu}$; this he calls the "H principle". The group chosen consists of the transformations π defined by $g_{\mu\nu}{}^\pi(x)=\pi_\mu{}^\alpha(x)g_{\alpha\beta}(x)\pi_\nu{}^\beta(x)$, $\det\pi>0$. Then the measure over $g_{\mu\nu}$-space becomes

$$\delta[g_{\mu\nu}]=N^{-1}\prod_x[-g(x)]^{-5/2}\prod_{\mu<\nu}dg_{\mu\nu}(x),$$

where N is a $g_{\mu\nu}$-independent normalization factor. {It seems to the reviewer that, if such a functional integral exists at all, this result follows merely from the requirement that $\delta[g_{\mu\nu}]$ be invariant under coordinate transformations. However, the author's contention that the measure in the Feynman quantization of any field is determined by such an H principle, regardless of the structure of the Lagrangian, is contradicted by nonlinear scalar theories such as $\mathscr{L}=f(\phi)\partial_\mu\phi\partial^\mu\phi$.} The operator field equations are shown to be

$$\langle 2|(R_\alpha{}^\beta-\tfrac{1}{2}\delta_\alpha{}^\beta R)(-g)^{\frac{1}{2}}|1\rangle=0.$$

With regard to (ii), the author argues, from the topological invariance of the theory, that the Hamiltonian must vanish identically. This is due to the fact that $\psi(f)$ must be independent of the parametrization of the hypersurface σ, and does not imply any lack of dynamic development in the ordinary sense. *P. W. Higgs.*

Husain, S. I. On unified field theory of gravitation and electromagnetism. II. Tensor (N.S.) **7** (1957), 81–85. [See also review below.]

The author considers the connection

$$\Gamma_{\lambda\mu}^{\nu}=\left\{\begin{matrix}\nu\\ \lambda\mu\end{matrix}\right\}+\delta_{\lambda}^{\nu}Q_{\mu}, \tag{1}$$

where $\left\{\begin{matrix}\nu\\ \lambda\mu\end{matrix}\right\}$ are Christoffel symbols of the "metric" tensor $h_{\lambda\nu}$ which has to satisfy

$$\nabla_{\mu}h_{\lambda\nu}=-2h_{\lambda\nu}Q_{\mu}. \tag{2}$$

Denoting by $R_{\omega\mu\lambda}^{\nu}$ the curvature tensor of $\Gamma_{\lambda\mu}^{\nu}$ and putting $R_{\omega\mu}\overset{\det}{=}R_{\omega\mu\lambda}^{\lambda}$, $\bar{R}_{\omega\mu\lambda}^{\nu}\overset{\det}{=}R_{\omega\mu\lambda}^{\nu}+n^{-1}R_{\omega\mu}\delta_{\lambda}^{\nu}$, one obtains at once the complete set of integrability conditions of (2):

$$\begin{aligned}&\text{a)}\quad \bar{R}_{\omega\mu\lambda}^{\alpha}h_{\alpha\nu}+\bar{R}_{\omega\mu\nu}^{\alpha}h_{\lambda\alpha}=0\\ &\text{b)}\quad (\nabla_{\xi_s}\cdots\nabla_{\xi_1}\bar{R}_{\omega\mu\lambda}^{\alpha})h_{\alpha\nu}+\nabla_{\xi_s}\cdots\nabla_{\xi_1}\bar{R}_{\omega\mu\nu}^{\alpha}h_{\lambda\alpha}=0,\end{aligned} \tag{3}$$

and Eisenhart's theorem applies [L. P. Eisenhart, Non-Riemannian geometry, Amer. Math. Soc. Colloq. Publ., v. 8, New York, 1927, pp. 14–18]. Of course $\Gamma_{\lambda\mu}^{\nu}$ and $\left\{\begin{matrix}\nu\\ \lambda\mu\end{matrix}\right\}$ have the same autoparallel lines.

{Remark of the reviewer: If the λ^{ν}-rank of the tensor set

$$R_{\omega\mu\lambda}^{\nu},\ \nabla_{\xi_s}\cdots\nabla_{\xi_1}R_{\omega\mu\lambda}^{\nu}\ (s=1, 2, \cdots)$$

is r, then there is a set of r linearly independent tensors $L_{\alpha\lambda}^{\nu}$, $\alpha=1, \cdots, r$ such that

$$R_{\omega\mu\lambda}^{\nu}=M^{\alpha}{}_{\omega\mu}L_{\alpha\lambda}^{\nu},$$
$$\nabla_{\xi_s}\cdots\nabla_{\xi_1}R_{\omega\mu\lambda}^{\nu}=M^{\alpha}{}_{\omega\mu\xi_s\cdots\xi_1}L_{\alpha\lambda}^{\nu}.$$

If

$$W_{\alpha\lambda}^{\nu}\overset{\det}{=}L_{\alpha\lambda}^{\nu}+n^{-1}\delta_{\lambda}^{\nu}L_{\alpha\alpha}^{\alpha},$$

then (3ab) for $s=1, 2, \cdots$ simplifies to

$$W_{\alpha\lambda}^{\alpha}h_{\alpha\nu}+W_{\alpha\nu}^{\alpha}h_{\lambda\alpha}=0.\}$$

V. Hlavatý (Bloomington, Ind.).

Treder, H. Zum Aufsatz von S. I. Husain "On unified field theory of gravitation and electromagnetism". Tensor (N.S.) **7** (1957), 128–129.

[See Husain, Tensor (N.S.) **6** (1956), 132–135; MR **18**, 543.] The author points out that Husain did not take into consideration that $|\mathfrak{h}|^{\frac{1}{2}}h^{\lambda\nu}V_{\lambda\nu}$ ($V_{\lambda\nu}\overset{\det}{=}R_{(\lambda\nu)\alpha}^{\alpha}$) contains also $h^{\lambda\nu}$ and therefore neglected the terms

$$|\mathfrak{h}|^{-\frac{1}{2}}\delta|\mathfrak{h}|^{\frac{1}{2}}h^{\lambda\nu}V_{\lambda\nu}/\delta h^{\omega\mu}=2k_{(\omega}{}^{\alpha}V_{\mu)\alpha}-\tfrac{1}{2}h_{\omega\mu}h^{\alpha\beta}V_{\alpha\beta}.$$

(The factor $1/\sqrt{|\mathfrak{h}|}$ is missing in the author's formula.) Moreover, the Husain theory leads to a covariant constant (with respect to $h_{\lambda\mu}$) electromagnetic field.

V. Hlavatý (Bloomington, Ind.).

Schmutzer, Ernst. Neue Interpretation der projektiven Relativitätstheorie und Anwendung auf ein kosmologisches Modell. Ann. Physik (7) **1** (1958), 136–144.

In a recent article, the author discussed a generalisation of 5-dimensional Riemannian geometry in the sense that the line-element $ds=e_{\nu}dx^{\nu}$ ($\nu=1, \cdots, 5$) no longer represents an exact differential [Z. Phys. **149** (1957), 329–339; MR **19**, 927]. The resulting formalism gives rise to a generalised projective theory of relativity. The present paper is concerned with further developments in this direction. It is conjectured that the variability of S (where $S^2=J=X_{\nu}X^{\nu}$) is connected with the expansion of the universe. A new set of field equations for a cosmological model are solved subject to: (1) the velocity of matter is sufficiently small to warrant neglect, (2) the electromagnetic field is negligibly small compared with the material field, (3) a spatial homogeneity of the mass distribution is assumed such that the field quantities S and g_{ik} ($i, k=1, \cdots, 4$) depend only on the time. Two distinct types of world-mass are considered. The theory gives rise to a Hubble constant which differs from that of earlier theories. Further cosmological implications are considered.

H. Rund (Durban).

Takeno, Hyôitirô. An addendum to "Some wave solutions of Einstein's generalized theory of gravitation" and "On some spherical wave solutions of non-symmetric unified field theories". Tensor (N.S.) **7** (1957), 141–142.

L'auteur indique que les solutions qu'il a obtenues [Tensor (N.S.) **6** (1956), 69–82, 90–103; MR **19**, 226, 367] pour la théorie unitaire non symétrique d'Einstein ont un $k_{\lambda\mu}$ de la "troisième classe" de la classification de Hlavatý.

Y. Fourès-Bruhat (Marseille).

See also: Fluid Mechanics, Acoustics: Khalatnikov.

Astronomy

Ogorodnikov, K. F. Statistical mechanics of the simplest types of galaxies. Astr. Ž. **34** (1957), 770–789. (Russian)

The model galaxies considered are characterized by three additive parameters, the mass, M, the sum of energies of motion of the stars, E, and the sum of the rotational moments of the stars, K. In accordance with principles given in a subsequent paper [Astr. Ž. **34** (1957), 809–819], the author derives the most probable phase distribution and finds that in the first approximation the system has an uniform rotation, "isothermal" distribution of the residual stellar velocities and a constant stellar density throughout the main body of the system. Since Poincaré's criterion for the upper limit of the rotational velocity of a fluid homogeneous sphere seems to be violated, the system is only "quasistationary". The volume in phase space is finite and the relaxation time smaller than the characteristic time scale of the system. The quasistationary systems are shown to expose the classical equilibrium figures of rotating fluids, the planetary and diskshape forms correspond to ellipsoide galaxies and spiral galaxies. The existence of needleshaped and pearshaped equilibrium figures is stressed.

In the second approximation, the additional star density is derived with the aid of a differential equation of Helmholtz' type. In the case of a diskshaped galaxy a Bessel equation is obtained for the radial additional density function, a harmonic solution for the azimuthal angle and an exponential decrease with the perpendicular distance from the plane of the disk. The zero order term, in which the azimuthal function is constant, already suffices to explain a system of rings like that of NGC 468, but for more complicated structures higher order terms come into consideration.

E. Lyttkens (Uppsala).

Mestel, L. Meridional circulation in shell-source stars. Astrophys. J. **126** (1957), 550–558.

Methods previously developed to study circulation through the radiative envelope of a Cowling-model star are applied to shell-source models. It is shown that (a) in the absence of a perturbing force, such as a centrifugal

field, any departure from spherical symmetry in the matter distribution in the radiative zone is rapidly wiped out, so that there is no tendency for spontaneous mixing to occur between the shell source and the rest of the star; (b) in order to drive mixing currents, an even higher rotation rate is needed than in a Cowling model; and (c) there is no reason to expect mixing to occur in binaries through the perturbing effect of one component on the other, except perhaps in cases where its effect is likely to be swamped by other evolutionary processes.

Author's summary.

Pickelner, S. B. The energy dissipation, heating and ionization of interstellar gas by shock waves. Astr. Ž. **34** (1957), 314–320. (Russian. English summary)

It is known that an increase of entropy takes place at a shock front. For weak shocks in an ideal gas, the amount of this increase is $-[\gamma(\gamma+1)/12T](\Delta V/V)^3$, where T is the temperature, ΔV is the change in the specific volume at the shock front and γ is the ratio of the specific heats. The generalization of this result when there are other contributing sources to the pressure and the internal energy is

$$\Delta S=-\frac{1}{12T}\left(\frac{\partial^2 P}{\partial V^2}\right)_S(\Delta V)^3.$$

In particular, for a perpendicular hydromagnetic shock (i.e., one in which the velocity of the wave is perpendicular to the magnetic field) the pressure P and the internal energy E are given by

$$P=p_{\text{gas}}+p_{\text{C.R.}}+H^2/8\pi,$$
$$E=\varepsilon_{\text{gas}}+2p_{\text{C.R.}}/\rho+H^2/8\pi\rho,$$

where the terms in $p_{\text{C.R.}}$ allow for the presence of cosmic rays. The corresponding expression for ΔS is

$$\Delta S=-\frac{1}{12T}\left[\gamma(\gamma+1)\frac{p_{\text{gas}}}{\rho}+6\frac{H^2}{8\pi\rho}+\frac{15}{4}\frac{p_{\text{C.R.}}}{\rho}\right]\left(\frac{\Delta V}{V}\right)^3.$$

This change in entropy leads to a dissipation of energy of amount $\Delta Q=T\Delta S$. The present paper investigates the astrophysical consequences of the dissipation of energy arising from this source. In particular, the author believes that by successive passages of a shock wave one may, under the conditions of interstellar space, increase the ionization of hydrogen by 20-50% and raise the mean temperature of the gas by 10,000°. *S. Chandrasekhar.*

Agostinelli, Cataldo. Piccoli movimenti in una massa gassosa stellare in equilibrio radiativo. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 347–352.

Kaplan, S. A. Shock waves in interstellar space. III. Gasomagnetic discontinuities. Astr. Ž. **34** (1957), 321–327. (Russian. English summary)

There is little new material in this paper, which is, however, a useful summary of previous work, and contains the main references. In order that the previous work may be extended to cover the interstellar gasomagnetic discontinuities, where there is considerable energy loss by radiation, the author relaxes the usual assumption of energy conservation during passage through the front; the temperatures are therefore given on both sides of the discontinuity.

By transforming to a system of coordinates in which the front is at rest, the author obtains very simple laws for the refraction and concentration of the lines of force, in the case when the strengths of both the shock and the field are large. The results are also given for the refraction problem of the interaction of two discontinuities inclined at a small angle.

The interaction of the magnetic with the flow field is greatest when these are mutually perpendicular, and this case is treated in some detail. Due to the "infinite" conductivity of the ionised material, the magnetic force lines are frozen in the fluid, so that the shock compression is accompanied by an increase in the field strength. The main effect of the field is to increase the pressure by an amount $H^2/8\pi$ and, if the gas pressure is small compared with the magnetic pressure, the compressibility produced by a shock is determined by the magnetic pressure and is given, for a strong shock, by

$$\rho_2/\rho_1=8\pi\rho_1V^2/H_1^2.$$

Because of the approximate equipartition of thermal, kinetic and magnetic energy in interstellar clouds, the magnetic pressure will generally be of the same order of magnitude as the gas pressure, and will therefore play an important role in the formation and collision of the clouds. Moreover, the author points out that in an ionisation front the local fields may be so high that the shock conditions are determined almost entirely by the magnetic pressure, and calculates the flow parameters in this case.

Finally, the author gives the automodel (similarity) solution for the propagation of a magnetic discontinuity, and shows that, in the plane case, this coincides with the steady state solution. The similarity variable is a dimensionless combination of the distance, time, field strength and density. *J. Hazlehurst* (Manchester).

Jaggi, R. K. On the stability of a gravitating sphere in a magnetic field. Proc. Nat. Inst. Sci. India. Part A. **23** (1957), 560–566.

The stability of an incompressible, gravitating, infinitely conducting fluid sphere is discussed for (a) Ferraro's [Astrophys. J. **119** (1954), 407–412; MR **16**, 183] case, where currents flow inside the sphere and the magnetic field is poloidal; and (b) De's [Z. Astrophys. **40** (1956), 21–27, MR **18**, 699] case, where the field is toroidal. Using the 'Energy Method', it is shown that for axisymmetric deformations of the form $rP_2(\cos\theta)$, the equilibrium configurations are stable. *K. C. Westfold.*

See also: Fluid Mechanics, Acoustics: Kahan; Piddington. **Optics, Electromagnetic Theory, Circuits:** Busbridge.

Geophysics

Krylov, Yu. M. Statistical theory and calculation of sea waves raised by wind. Trudy Gos. Okeanograf. Inst. **33(45)** (1956), 5–79. (2 plates) (Russian)

Le mémoire, qui ne représente qu'une première partie d'un travail plus vaste, est consacré au développement d'une théorie statistique du problème des vagues provoquées par le vent.

Le premier chapitre est consacré à une étude critique des travaux théoriques et expérimentaux, surtout de l'école russe. L'auteur passe en revue les méthodes de mesures des différents éléments de la vague. Il fait une étude assez détaillée des formules empiriques liant les différents paramètres en fonction de la vitesse du vent. Suit une étude des différents instruments de mesure.

L,auteur expose les travaux statistiques de Titoff, Zeliouk, Wilensky et Glouchovsky.

Le premier essai d'une étude théorique pour expliquer les courbes de distribution empirique a été tenté par l,auteur en utilisant les rèsultats de Rayleb sur la fonction de distribution de l'amplitude résultant d'une oscillation harmonique. Suit une étude des travaux de Makkaveeff, basée sur l'application de l'équation de l'énergie ainsi que les travaux de Schouleikine.

Dans le deuxième chapitre l'auteur expose sa théorie statistique. Il introduit comme base l'étude des vagues à 3 dimensions comme elles se présentent dans la nature. Leurs éléments géométriques sont la hauteur (suivant la verticale réunissant le point le plus haut au point le plus bas), la longueur, l'étendue de la crête et la direction de la propagation. Par des raisonnements assez élémentaires l'auteur déduit la fonction de densité de probabilité de r (le rayon vecteur):

$$(1)\quad f(r)=\frac{\pi}{2\bar{r}}\frac{r}{\bar{r}}\exp\left(-\frac{\pi}{4}\left(\frac{r}{\bar{r}}\right)^2\right)\ (\bar{r}\text{ la valeur moyenne}),$$

qui est une distribution normale à 2 dimensions.

La même distribution est valable pour les hauteurs des vagues, en remplaçant dans (1) $r/\bar{r}$ par $h/\bar{h}$.

L'auteur déduit ensuite la loi de distribution des hauteurs des vagues à 3 dimensions et cherche une liaison statistique entre les altitudes des vagues à 3 dimensions et les altitudes en chaque point.

Il en déduit ensuite les fonctions de distribution des périodes ainsi que les vitesses de propagation des ondes.

Les mesures expérimentales des différents paramètres de l'onde confirment assez bien les résultats obtenus par l'auteur. Ce dernier étudie, après quelques applications pratiques des résultats obtenus, l'utilisation des fonctions de distribution pour étudier les inscriptions des traceurs des vagues.

Le chapitre 3 est consacré à l'étude des moyennes des éléments des vagues. A cet effet l'auteur fait une étude détaillée de la cinématique et de la dynamique des ondes à 3 dimensions ainsi qu'une étude de l'équation d'énergie.

Suit une étude des relations qui existent entre les différents paramètres de la vague.

Enfin l'étude est accompagnée d'une bibliographie assez détaillée, principalement en ce qui concerne les travaux russes. *M. Kiveliovitch* (Paris).

★**Pierson, Willard J., Jr. On the phases of the motions of ships in confused seas.** Research Division, College of Engineering, New York University, New York, Tech. Rep. No. 9, Nonr 285 (17), 38 pp. (1957).

The author uses a random process as a model of the sea surface [see Pierson, Advances in Geophysics **2** (1955), 93–178] and computes the cross-spectra between the heaving, pitching and rolling motions of a ship and the forcing seaway. The discussion is an elaboration of that given by M. St. Denis and the author [Trans. Soc. Navak. Arch. Marine Engrs. **61** (1953), 280–357]. *M. Rosenblatt* (Bloomington, Ind.).

'Arnason, Geirmundur. A convergent method for solving the balance equation. J. Meteorol. **15** (1958), 220–225.

The non-linear balance equation can be written in the form

$$f\nabla^2\psi+2[\psi_{xx}\psi_{yy}-\psi_{xy}^2]+\nabla f\cdot\nabla\psi-\nabla^2\phi=0,$$

where ψ is the stream function, f the coriolis parameter and ϕ the geopotential. A method for solving this equation for ψ by using successive approximations is described, and the conditions governing convergence are stated; these are shown to be the conditions for ellipticity of the balance equation. A slightly different method of solution is shown to diverge, and an example is given. *M. H. Rogers.*

de Jong, H. M. Errors in upper-level wind computations. J. Meteorol. **15** (1958), 131–137.

An analysis is made of the random errors involved in the evaluation of upper wind data, and there is a brief discussion of determinative errors. *M. H. Rogers.*

Musaelyan, Š. A. On centers of activity of the atmosphere. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1957**, 1520–1523. (Russian)

Sakagami, Jiro. On the atmospheric diffusion of gas or aerosol near the ground. Nat. Sci. Rep. Ochanomizu Univ. **7** (1956), 25–61.

Camps, F. Influence des erreurs des distances principales en aérotriangulation et aéronivellement. Bull. Soc. Roy. Sci. Liège **26** (1957), 142–157.

The principal distances of the projectors in a reduction instrument, as the stereoplanigraph, are never strictly equal, nor strictly equal to the principal distance of the picture taking camera. The author studies the influence of the resulting errors on the relative orientation of a pair of plates and on an aerial triangulation. For this purpose he takes up again the theory of errors of relative orientation and establishes the laws of propagation of these errors in aerial triangulation taking into account the errors $\Delta f'$ and $\Delta f''$ introduced through the two principal distances of the projectors of the reduction instrument. He establishes that in aerial triangulation, the influence of the errors $\Delta f'$ and $\Delta f''$ is limited to an error $(Z/f)(\Delta f''-\Delta f')$ affecting systematically all the elements bz of odd order. All the other elements of the triangulation are independent of $\Delta f'$ and $\Delta f''$. He presents the results of a controlled experiment which confirm the theoretical results. *P. D. Thomas.*

Hristow, Wladimir K. Tafeln zur Maschinenrechnung der Gaussschen Koordinaten, der Meridiankonvergenz und des Massstabes aus den geographischen Koordinaten auf dem Ellipsoid von F. N. Krassowsky für alle Breiten. Bŭlgar. Akad. Nauk. Trudove Central. Lab. Geod. 1956, 228 pp. (Russian, Bulgarian and German text)

The author's book, Die Gausschen und geographischen Koordinaten auf dem Ellipsoid von Krassowsky [Verlag Technik, Berlin, 1955; MR **17**, 1236], gave a comprehensive theoretical account of the development of the Transverse Mercator (Gauss) projection. These tables can be considered as a valuable practical addition to the book, by presenting means for the numerical computation of grid coordinates, meridian convergence, and scale factor. The geodetic reference surface employed is the Krassovsky ellipsoid. It is this which makes these tables unique, since similar tables to the same order of accuracy have been previously published by the United States Army Map Service (AMS) for a number of other geodetic reference ellipsoids.

Formulas for the computations are of the form (*) $\sum a_i(\phi)l^i$, where ϕ is the geographic latitude, and l is a suitably scaled difference of longitude. The tables consist of listings of a_i for $\phi=0°(1')90°$. (The AMS tables are limited to $\phi=0°(1')80°$.) These yield an accuracy of 10^{-3} meter for grid coordinates, 10^{-3} second of arc for meridian convergence, and 1 part in 10^8 for the scale factor.

In the introduction to the tables, the author also presents formulas of the form $\sum\sum a_{ij}\phi^i\psi^j$, which are true power series with constant coefficients based on a specified origin. It is pointed out that series of this type are better suited for high-speed electronic computers than those of type (*). The latter, however, are superior for desk calculator computation (provided tables such as these are available), since far less terms are involved.

B. Chovitz (Washington, D.C.).

Bragard, L. Remarque sur une équation aux dérivées partielles de la géodésie. Bull. Soc. Roy. Sci. Liège **26** (1957), 131–135.

The author considers the solution, by the method of Picard, of a partial differential equation of elliptic type, satisfied by the difference of major curvatures of the principal sections of two surfaces at corresponding points having the same polar coordinates. The two surfaces differ respectively from the ellipsoid of revolution $r_E = a_m(1+\sum_{i=1}^{I} A_i X_{2i})$, where X_{2i} is the Legendre polynomial of degree $2i$, by the additive functions $a_m\sum_{n=2}^{\infty} u_n$ and $a_m\sum_{n=2}^{\infty} v_n$, where u_n, v_n are spherical harmonics of a surface of degree n. The author applies an integral property of spherical harmonics and obtains an integral equation of Fredholm type with logarithmic kernel which is equivalent to the elliptic partial differential equation. The author refers to the results of an unpublished paper in obtaining the elliptic partial differential equation and in solving the Fredholm integral equation. *P. D. Thomas.*

Camps, F.; et Lierneux, L. Influence des erreurs de centrage des clichés en aérotriangulation et aéronivellement. Bull. Soc. Roy. Sci. Liège **26** (1957), 189–204.

Since it is physically impossible to mount a plate in the projector of a photoreduction instrument (as the stereoplanigraph) in a position identical to that occupied by the film in the camera at the time the pictures were taken, errors result which are called plate centering errors. The author concludes from a theoretical and experimental study that these errors are compensated by the displacements of the projector during the operations of obtaining relative orientation and scale transmission, and therefore have no influence on the coordinates of a survey point in the course of an aerial triangulation. However, the plate centering errors appear in the readings of the counters on the height and coordinate movements of the photoreduction instrument. *P. D. Thomas.*

★ **Dupuy, Michel. L'interpolation complexe et ses applications en géodésie et cartographie.** Thèses, Université de Paris, Institut Géographique National, Paris, 1954. 101 pp.

A résumé of this thesis was listed in MR **15**, 163.

See also: Numerical Methods: Gol'cman. **Fluid Mechanics, Acoustics:** Eckart.

OTHER APPLICATIONS

Economics, Management Science

★ **Beach, E. F. Economic models: an exposition.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1957. xi+227 pp. $7.50.

Part One reviews economic theories expressed as systems of equations — generally, differential or difference equations — which are supposed to be approximately satisfied by such variables as quantities demanded or supplied, and prices offered or asked at a given time. In Part Two, random disturbances are introduced and the statistical estimation of their distribution and of the coefficients of the equations is discussed. The book is written for students of economics who start with practically no mathematical background, but are presumably helped to acquire one as they study the examples and applications offered in the book. *J. Marschak* (New Haven, Conn.).

Luce, R. Duncan. A probabilistic theory of utility. Econometrica **26** (1958), 193–224.

One of the attractive features of this interesting paper is the way in which its hypotheses are motivated by the general character of experimental results. Most non-probabilistic theories require that choices between pairs of alternatives (on the basis of preference or loudness or weight etc.) be consistent and transitive; yet experimental facts often violate these conditions. Inconsistency seems to call for a theory which yields only a probability that a will be chosen over b, and this in turn will permit intransitivity.

The basic concept is that of a discrimination structure on a set S, i.e., a nonnegative real-valued function $P(a, b)$ on $S\times S$ such that $P(a, b)+P(b, a)\leqq 1$, and such that there is at least one distinct pair a, b for which $P(a, b)\neq P(b, a)$. $P(a, b)$ is interpreted as the probability that the subject will choose a over b. A discrimination structure generates a relation $\gtrsim$, the trace of P; $a\gtrsim b$ holds if $P(a, c)\geqq P(b, c)$ and $P(c, a)\leqq P(c, b)$ for every c in S. The trace is obviously reflexive and transitive. Most of the results require the further assumption that it is connected, and hence a weak order.

Various properties of discrimination structures and their traces are studied, together with their plausibility in the preference context. In the usual way, S is extended to include probability mixtures: let α be a member of a Boolean algebra of events, then $a\alpha b$ is the lottery ticket which gives a if α occurs and b if it does not. Choices among lottery tickets are assumed to obey the usual "natural" rules, to make them independent of the number of "stages", for instance.

Let $Q(\alpha, \beta)$ be the objective probability that the subject (subjectively) judges α more likely to occur than β. A discrimination structure on a set of pure alternatives and lottery tickets is said to be decomposable if the activity of discriminating between outcomes and the activity of making subjective probability judgments are statistically independent, i.e., if $P(a\alpha b, a\beta b) = P(a, b)Q(\alpha, \beta) + P(b, a)Q(\beta, \alpha)$. This is a strong assumption and underlies most of the results.

I am still several definitions away from being able to quote any of the more important theorems, so I forebear. The results include some strong restrictions on the form of Q and on the form of a von Neumann-Morgenstern utility function, if one exists. *R. Solow* (Cambridge, Mass.).

Stone, Richard; and Rowe, D. A. The market demand for durable goods. Econometrica **25** (1957), 423–443.

A dynamic theory of demand for durable and semi-durable goods is developed. An investment function is introduced, reducing the gap between the opening stock

and the equilibrium stock. The theory is applied to data covering a prewar and a postwar period for which a satisfactory common relationship could be obtained.

S. Malmquist (Stockholm).

Ammeter, Hans. Die Ermittlung der Risikogewinne im Versicherungswesen auf risikotheoretischer Grundlage. Mitt. Verein. Schweiz. Versich.-Math. **57** (1957), 145–200.

From the standpoint of the collective risk theory, the author discusses various ways of building up the graduation reserves. The usual basis of the risk theory, a fixed probability of loss and a constant distribution of losses, is generalized by using the Polya-Eggenberger distribution. *P. Johansen* (Gentofte).

Grenander, Ulf. On heterogeneity in non-life insurance. I. Skand. Aktuarietidskr. **40** (1957), 71–84.

Bei allgemeinen Risikoversicherungen wird die Wahrscheinlichkeit p_k einer Police, k Schäden in einer bestimmten Zeitspanne zu erleiden, dargestellt durch

$$p_k = \int_0^\infty e^{-\lambda} \frac{\lambda^k}{k!}\, d\mathfrak{G}(\lambda).$$

λ wird als der Risikoparameter bezeichnet, $\mathfrak{G}(\lambda)$ ist eine im allgemeinen nicht bekannte Verteilungsfunktion. Der Verf. gibt in diesem ersten Teil Schätzungen für den Risikoparameter einer gegebenen Police und $\mathfrak{G}(\lambda)$ sowie Teste für die Schätzung der Inhomogenität einer Gruppe von Policen. *W. Saxer* (Zürich).

Grenander, Ulf. On the theory of mortality measurement. II. Skand. Aktuarietidskr. **39** (1956), 125–153 (1957).

[For part I see Skand. Aktuarietidskr. **39** (1956), 70–96; MR **19**, 188.] The second part discusses the relation between parametric and non-parametric mortality measurements in terms of validity and efficiency. It is shown how a priori information can be used to obtain estimates of extra high precision. Some numerical calculations show that the non-parametric method gives highly efficient results compared with the standard Makeham graduations. *P. Johansen* (Gentofte).

Lovera, Piera. Sopra alcune diseguaglianze che si presentano nella matematica attuariale. Giorn. Ist. Ital. Attuari **19** (1956), 131–139.

The author proves several elementary inequalities involving decreasing sequences of positive numbers and also some inequalities for annuities. He states that his relations (which are too complicated to be listed here) are similar to, but sharper than, certain inequalities given by G. Mantellino in a paper [Ist. Mat. Finanz. Univ. Torino, 1953] not accessible to this reviewer. *E. Lukacs.*

Arfwedson, G. Notes on collective risk theory. Skand. Aktuarietidskr. **40** (1957), 46–59.

Ergänzende Bemerkungen zu den Abhandlungen des Verf., Cramér und Ammeter über Risikotheorie und insbesondere das Ruinproblem betreffend die folgenden Punkte: Existenztheoreme; Integralgleichungen der Risikotheorie; Fourier-Transformierte; erste Ruin-Zeitpunkte. *W. Saxer* (Zürich).

Johnson, N. L. Uniqueness of a result in the theory of accident proneness. Biometrika **44** (1957), 530–531.

Castoldi, Luigi. Alternanza di code e regolazione del traffico in un incrocio stradale. Rend. Sem. Fac. Sci. Univ. Cagliari **27** (1957), 48–64.

The behavior of traffic at an intersection is studied non-probabilistically, that is, using mean values of density, velocity, etc., as if they were constants. The simple case of a one-way pedestrian lane across a one-way vehicular street is analyzed by supposing (say) a pedestrian waiting queue accumulating at a constant rate while an already accumulated queue of vehicles moves across the lane. When the queue of vehicles has finished, the pedestrian queue moves across the street and a new queue of vehicles accumulates. This process continues in see-saw fashion. Simplifying assumptions are made, which reduce the behavior to repeated solutions of a linear equation and an inverse quadratic equation respectively, and this leads to a simple condition for convergence or indefinitely increasing delays. More complex cases are studied in similar fashion, for example, behavior at an intersection traversed by two two-way traffic streams and two two-way pedestrian streams, all under the control of a traffic light that permits one of three states: *N-S* traffic only, *E-W* traffic only, and pedestrian crossings only.

W. Kruskal (Chicago, Ill.).

Gordon, Robert. Optimum component redundancy for maximum system reliability. Operations Res. **5** (1957), 229–243.

See also: Probability: Reich; Jackson; Stephan.

Programming, Resource Allocation, Games

Bellman, Richard; and Dreyfus, Stuart. Dynamic programming and the reliability of multicomponent devices. Operations Res. **6** (1958), 200–206.

A simple example illustrates the use of the "Lagrange multiplier" technique for solving problems of the type indicated in the title, where types and quantities of components at various stages are chosen subject to given cost and weight constraints. *J. Kiefer* (Oxford).

Bellman, Richard; and Dreyfus, Stuart. On a tactical air-warfare model of Mengel. Operations Res. **6** (1958), 65–78.

This is the first of a series of papers devoted to the computational solution of dynamic programming processes. In it we use the functional-equation approach to treat a tactical air-warfare model that A. Mengel [unpublished] previously has considered by means of classical variational techniques. (Authors' summary)

J. Kiefer (Oxford).

Bellman, Richard. Notes on the theory of dynamic programming. Transportation models. Management Sci. **4** (1958), 191–195.

This paper illustrates some applications of the functional equation technique of the theory of dynamic programming to a general class of problems arising in the study of networks, particularly those arising in transportation theory and including the Hitchcock-Koopmans transportation problem.

M. M. Flood (Ann Arbor, Mich.).

McCloskey, Joseph F.; and Hanssmann, Fred. An analysis of stewardess requirements and scheduling for a major domestic airline. Naval Res. Logist. Quart. **4** (1957), 183–191.

Friedman, Lawrence; and Yaspan, Arthur J. An analysis of stewardess requirements and scheduling for a major airline. Annex A: The assignment problem technique. Naval Res. Logist. Quart. **4** (1957), 193–197.

Friedman, Lawrence. An analysis of stewardess requirements and scheduling for a major airline. Annex B: Calculation of the school table. Naval Res. Logist. Quart. **4** (1957), 199–202.

These three papers, main and two annexes, report a team result. The main paper shows how a complex airline stewardess training and scheduling problem was approximated mathematically by a combination of a linear programming model and an inventory control model. In Annex A, Friedman and Yaspan outline a procedure (without proof) for finding a minimal set of lines (rows and columns) that includes all the zero elements in a square matrix of non-negative real numbers; they show, by numerical example, how this procedure can be used in Kuhn's Hungarian Method to solve the assignment problem of linear programming. The Friedman-Yaspan procedure includes a random choice among available zeros, at one stage, that makes it very probable that their algorithm will terminate rapidly; however, Kuhn (and several others) have given effective algorithms for this problem that are certain to terminate in a finite number of steps. In Annex B, Friedman finds an approximate solution for the minimal value of the shortage-surplus cost function, represented by a triple sum of double integrals, by a combination of elementary mathematical analysis and sensible assumptions about the practical situation.

M. M. Flood (Ann Arbor, Mich).

Ford, L. R., Jr.; and Fulkerson, D. R. A simple algorithm for finding maximal network flows and an application to the Hitchcock problem. Canad. J. Math. **9** (1957), 210–218.

Let the set of nodes $P_1, \cdots, P_n$ of a finite graph be given, and for $i, j=1, \cdots, n$ let c_{ij} be a nonnegative integer denoting the capacity of the directed arc (P_iP_j). The problem of determining the maximal flow (of an homogeneous commodity conserved at nodes) through the graph from the source P_1 to the sink P_n is that of finding x_{ij} such that $0 \leq x_{ij} \leq c_{ij}$ $(i, j=1, \cdots, n)$, $\sum_{j=1}^{n}(x_{ij}-x_{ji})=0$ $(i=2, \cdots, n)$, and such that $\sum_j x_{1j}$ is maximized.

The authors presented earlier [same J. **8** (1956), 399–404; MR **18**, 56] a combinatorial algorithm for solving this problem in the case of a planar graph. This paper gives an algorithm for the case of a general graph, and employs it in a further algorithm for the solution of the ordinary transportation problem: Minimizing the "cost" $\sum_{i,j} C_{ij}x_{ij}$ of flow through a graph whose nodes consist only of sources and sinks having specified flows and which has only arcs of the form (P_iQ_j), where P_i is a source, Q_j a sink, C_{ij} is the cost of transportation from P_i to Q_j, and x_{ij} is the corresponding flow to be determined.

Proofs for these procedures are given here in full; a detailed description of the efficient use of the transportation algorithm is given elsewhere [Management Sci. **3** (1956), 24–32]. This procedure is now the most commonly used for problems of this class. *P. Wolfe.*

Prager, William. Numerical solution of the generalized transportation problem. Naval Res. Logist. Quart. **4** (1957), 253–261.

A four-source, five-sink capacitated transportation problem in which one source and one sink are required to have stipulated shadow prices is solved by the procedure of Ford and Fulkerson [same Quart. **4** (1957), 47–54; MR **19**, 619], modified by forbidding changes of the stipulated prices. *P. Wolfe* (Santa Monica, Calif.).

Munkres, James. Algorithms for the assignment and transportation problems. J. Soc. Indust. Appl. Math. **5** (1957), 32–38.

A variant of the Hungarian method for the optimal-assignment problem, differing from that of Kuhn [Naval Res. Logist. Quart. **2** (1955), 83–97; MR **17**, 759] in the procedure for finding a minimal cover for a set of matrix entries, is given. An upper bound for the number of "elementary operations" ("scan a line", etc.) of the order of n^3 is proved for the solution of an n-by-n assignment problem with the author's algorithm. A generalization of the algorithm to the solution of the transportation problem is given. *P. Wolfe* (Santa Monica, Calif.).

Frisch, Ragnar. The multiplex method for linear programming. Sankhyā **18** (1957), 329–362.

The author formulates a linear programming problem: $n+m$ real variables satisfy m linearly independent equations. There are n degrees of freedom. Let $x_u, x_v, \cdots, x_w$ be a basic set of variables. The equations are expressed in terms of this basic set: $x_j=b_{j0}+\sum b_{jk}x_k$ $(j=1, 2, \cdots, n+m, k$ in basic set). The preference function is: $f=p_0+\sum p_kx_k$. The solutions cannot be negative: $x_j \geq 0$ $(j=1, 2, \cdots, n+m)$. An initial point is determined which satisfies the equations, and a movement in the preference direction is defined as $x_k=x_k^0+\lambda p_k$, where x_k^0 are the coordinates of the initial point. Variables not in the basic set are given by $x_j=x_j^0+d_j$ $(d_j=\sum b_{jk}p_k)$. The largest permissible value of the parameter λ is found. This gives the breaking out point. But positive values of λ which are greater than the permitted value are also considered. They are the optimum candidates and it is claimed that at each round of computations the additional candidates should equal the square root of the number of degrees of freedom, which is realised at the breaking out point. Various other procedures are discussed. This method provides an alternative to the simplex method and might shorten computations considerably by shrewd guessing. *G. Tintner* (Ames, Iowa).

★ **Karlin, Samuel; and Restrepo, Rodrigo. Multistage poker models.** Contributions to the theory of games, vol. 3, pp. 337–363. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

Several poker models are described, and the corresponding continuous game is explicitly solved in each case. The two players A and B ante and independently receive "hands" x and y, respectively, each hand being a number in the unit interval chosen according to the uniform distribution (or some other given distribution). The larger hand wins. Both the case where any one of the amounts $a_1<a_2<\cdots<a_n$ can be bet by A, with no raises, and the case where only one amount a can be bet with k raises, are treated. In the first case, for example, optimal strategies are given in terms of explicitly determined numbers $b_i, c_i, i=1, \cdots, n, b_i<b_{i+1}, c_i<c_{i+1}, b_1=c_1$, as

follows. For A: Bet a_t if $c_t \leqq x < c_{t+1}$; if $x < c_1$ bet a_t with probability $\phi_t(x)$ arbitrary, subject to the constraints $\sum_{t=1}^{n} \phi_t(x) \leqq 1$, $\int_{c_t}^{c_{t+1}} \phi_t(x)dx = c_1(c_{t+1} - c_t)$. For B: If A bets a_t, then call if $y \geqq b_t$ and fold if $y < b_t$. Interest in these models comes partly from similarities in method with statistical decision problems. *W. H. Fleming.*

Zoroa, P. Applications of the theory of games. Trabajos Estadist. **9** (1958), 21–42. (Spanish)

An expository paper based on the books of Blackwell and Girshick [Theory of games and statistical decisions, Wiley, New York, 1954; MR **16**, 1135] and J. C. C. McKinsey [Introduction to the theory of games, McGraw-Hill, New York, 1952; MR **14**, 300]. The applications are to games of the statistician versus nature. *J. M. Danskin.*

Zoroa, Procopio. A note on the definition of equivalent games. Trabajos Estadist. **9** (1958), 13–20. (Spanish)

The book "Theory of games and statistical decisions" [Wiley, New York, 1954; MR **16**, 1135], by Blackwell and Girshick, contains a definition of the equivalence of two games. The author gives a generalized definition of the equivalence of two games, which avoids, he states, the "chain of reductions whose determination may not be simple". He obtains the expected properties of this equivalence. *J. M. Danskin* (New Brunswick, N.J.).

★**Isbell, J. R. Finitary games.** Contributions to the theory of games, vol. 3, pp. 79–96. Annals of Mathematics Studies, no. 39. Princeton University Press, Princeton, N. J., 1957. $5.00.

The author presents a variety of results on the existence of equilibrium points in certain classes of strategies for certain classes of games. Introducing strategies which are mixtures of behavior strategies, he shows that the usual restriction that a play of a game may not meet an information set more than once can be removed and the game will still have a solution. The author also proves that "programming games" whose payoff function is the quotient of two multilinear forms have equilibrium points. *M. Dresher* (Santa Monica, Calif.).

Reichbach, M. Ein Spiel von Banach und Mazur. Colloq. Math. **5** (1957), 16–23.

In this game the players choose in alternation the strictly positive strictly decreasing terms of an infinite series; the first player tries to make the series converge to a sum s in a specified set M. The author gives an example to show that the problem has little to do with topological or measure-theoretic concepts (the first player can have a forced win although M is compact nowhere-dense of measure 0) and proceeds to formulate conditions on M stronger than being of the first category which guarantee a win for the second player. *J. Isbell.*

See also: Linear Algebra: Goldberg. **Topological Vector Spaces:** Fan. **Numerical Methods:** Riley and Gass. **Miscellaneous:** Goodell.

Biology and Sociology

Leslie, P. H. A stochastic model for studying the properties of certain biological systems by numerical methods. Biometrika **45** (1958), 16–31.

A deterministic model for the growth of a single species, or the interaction of two species, can be paralleled by a stochastic model in which $N_{t+1,i}$, the population size of the ith species at time $t+1$, develops by a birth and death process, the values of $N_{t,j}$ ($j=1, 2$) being given. Since the expectation and variance of $N_{t+1,i}$ can be expressed in terms of t, the $N_{t,j}$ and the parameters of the model, realisations of the stochastic process can be obtained approximately by a series of random (e.g. normal) deviates. Details are given for (a) the logistic growth of a single species, (b) two competing systems, and (c) the predator-prey model. For a given deterministic formulation of the logistic model, there is a series of stochastic analogues, bounded by the situations where the birth-rate or the death-rate is constant. For two sets of parameters, a set of 10 realisations is reported for each of the extreme situations; the stochastic means do not differ greatly from their deterministic equivalents. Finally, the author points out that the chance of extinction in a finite time, for a logistic population, will in certain circumstances be exceedingly small. *P. Armitage.*

George, F. H. Logical networks and probability. Bull. Math. Biophys. **19** (1957), 187–199.

Information and Communication Theory

Schützenberger, Marcel P. On the quantization of finite dimensional messages. Information and Control **1** (1958), 153–158.

L is a measure of quantization noise, and H is the negentropy of the quantized form of an n-dimensional continuous signal; intuitively, L and H are roughly reciprocal, and the author gives a lower bound to L as a function of H, for certain natural L. *V. E. Beneš.*

Good, I. J.; and Doog, K. Caj. A paradox concerning rate of information. Information and Control **1** (1958), 113–126.

The authors define a "Riemannian" rate of information, yielded by a "received" stochastic process $R(t)$ about a "transmitted" stochastic process $S(t)$, by first computing the information per unit time

$$\frac{1}{T} E\left\{\log \frac{P[S(t_i)\&R(t_j)]}{P[S(t_i)]P[R(t_j)]}\right\} (0 = t_0 < t_1 < \cdots < t_n = T),$$

then letting $\max |t_i - t_j| \to 0$, and finally $T \to \infty$. The limit, if it exists, is denoted by J. It should be pointed out that J was defined in essentially the same way by Pinsker [Dokl. Akad. Nauk. SSSR (N.S.) **99** (1954), 213–216; MR **16**, 495]. An explicit computation of J is made for the case when $R(t) = S(t) + N(t)$, where $S(t)$ and $N(t)$ are mutually independent stationary Gaussian processes. In the example (*) when $S(t)$ and $N(t)$ both have spectral density $F'(w)$ constant over $[0, W]$, and vanishing outside $[0, W]$ (so that S and N are both singular), it turns out that $J = \infty$. (Pinsker's results imply that $J = \infty$ if S and N have identical spectral densities and are regular.) The authors are disturbed that J can be infinite, and suggest some possible explanations. However, the main paradox arises from the fact that according to an intuitive invariance principle (cf. e.g. Shannon, [The Mathematical theory of communication, Univ. Illinois Press, Urbana, 1949; MR **10**, 133; **11**, 258], the reviewer [J. Math. Phys., **30** (1951), 156–161; MR **14**, 568], Gel'fand and Yaglom [Uspehi Mat. Nauk (N.S.) **12** (1957), no. 1 (73), 3–52; MR **18**, 980]), which has never been stated in a precise, sufficiently general way, J should be invariant under a

transformation preserving the "dimension rate" of R and S. If one assumes that the invariance principle applies to (*), it follows that $J<\infty$.

The theorem regarding the nonvanishing of $\det[\rho(t_i-t_j)]$ if the spectral function is absolutely continuous is correct, but the proof given has a gap.

{The name of the second author is understood to be a pseudonym.} *E. Reich* (Minneapolis, Minn.).

Melton, Ben S.; and Karr, Philip R. Polarity coincidence scheme for revealing signal coherence. Geophysics **22** (1957), 553–564.

A simple statistical procedure, based on multiple coincidence of field polarities as indicated by several receptors, offers a criterion of the existence of a signal below the general noise level. As an example, with six receptors in a field where there is a sine wave signal ten decibels below the noise level, there will be twice the number of coincidences that would be obtained for the noise alone. A scheme is shown to illustrate how an electrical system can perform the analysis and present the results continuously. *Authors' summary.*

Brown, A. F. R. Language translation. J. Assoc. Comput. Mach. **5** (1958), 1–8.

A technique of machine translation from French into English is described, although the author states that the method is applicable to other languages. For each French word there is a sequence of instructions to be obeyed in order according to priority number. These instructions are to be found a) in a general list applicable to every sentence, or b) from a dictionary stored in the machine, giving instructions along with the English equivalent for the word. The instructions themselves are compounded out of 48 basic suboperations stored in the high speed memory. These suboperations include the ability to delete words, change order, look for certain grammatical characteristics, etc. When the instructions for all the words of a sentence have been completed, a translation is available. The novelty of the approach described here is that the instructions have been evolved experimentally by developing them as required to translate a sequence of French sentences rather than by a priori consideration of the structures of French and English. At the time of writing, a computer program had been written but it had not been tried on any new text. *C. C. Gotlieb.*

Control Systems

Gopp, Yu. A. Application of D-decomposition for the construction of characteristics of roots and the investigation of the quality of systems of automatic regulation. Avtomat. i Telemeh. **17** (1956), 789–798, appendix to no. 9, 1. (Russian. English summary)

The operation of the D-decomposition is used in the construction of families of curves which are the images of straight lines parallel to the imaginary axis. This makes it possible to calculate the character of the roots which determine the curved variations of the real and imaginary parts of the characteristic equation in the function of the variable parameters, and to establish optimum values for the parameters which insure the desired quality of the transient process. The method is illustrated by a numerical example.

This work analyzes and extends the work of Cypkin and Bromberg [Izv. Akad. Nauk SSSR Otd. Tehn. Nauk **1945**, 1163–1168; MR **7**, 519], Fel'dbaum [Avtomat. i Telemeh. **9** (1948), 253–279; MR **10**, 119], Rubinčik ["A graphical method for determining the domain of distribution of characteristic values of equations", Regulation of machines and synthesis of mechanisms, Mašgiz, 1950; "An approximate method for estimating control quality in linear systems", Organization and elements of the theory of automation and telemechanics, Mašgiz, 1952], Semenov [Trudy Sem. Teorii Mašin i Mehanizmov **6** (1949), no. 43, 5–29; MR **13**, 292], Myasnikov [Avtomat. i Telemeh. **10** (1949), 267–273; MR **11**, 103], Kuzovkov [Naučno-Tehn. Sb. NII MPSS no. 1 (1949)], Uderman [Električestvo **1952**, no. 2, 59–67], Neĭmark [Avtomat. i Telemeh. **9** (1948), 190–203; Stability of linear systems, Izdat. LKVVIA, 1949; MR **12**, 498], Popovskiĭ [Avtomat. i Telemeh. **14** (1953), no. 3], Tolle [Die Regelung der Kraftmaschinen, 3rd ed., Springer, Berlin, 1921], Železcov [Avtomat. i Telemeh. **10** (1949), 424–436; MR **11**, 597], and Bulgakov [Dokl. Akad. Nauk SSSR (N.S.) **73** (1950), 1143–1144; Prikl. Mat. Meh. **14** (1950), 453–458; MR **12**, 498, 335]. *S. D. Zeldin.*

Adams, K. M. On the synthesis of three-terminal networks composed of two kinds of elements. Philips Res. Rep. **13** (1958), 201–264.

A set of necessary and sufficient conditions and a method of realization of all sets of series-parallel LC three-terminal network functions from the zeroth to the sixth degree are given. The conditions are necessary for the synthesis of series-parallel LC three-terminal networks, but it is not known whether they are also necessary for the synthesis of three-terminal networks of arbitrary structure. *J. Hartmanis* (Schenectady, N.Y.).

Brown, Wm. M. Analysis of discrete linear systems. J. Soc. Indust. Appl. Math. **5** (1957), 206–224.

Having in view applications to the theory of control systems operating with discrete data, the author discusses linear difference equations with constant coefficients, particularly equations of the form

$$g(n)=\sum_{k=-\infty}^{\infty} W(n-k)[f(k)-g(k)],$$

where W and f are known functions and g is the unknown. The form in which the theory is developed is designed to emphasize analogies with the theory of control systems operating with continuous data. Instead of the Laplace transforms used in the latter theory, the author uses here what he calls Z-transforms, the Z-transform of a sequence $\cdots, f(-1), f(0)\ f(1), \cdots$ being the function $F(z)=\sum_{n=-\infty}^{\infty} f(n)z^n$. Sequences $\cdots, f(-1), f(0), f(1), \cdots$ in which the $f(k)$ are random variables are considered, and the appropriate statistical concepts, e.g., the concept of a power spectrum, are developed in terms of Z-transform theory. Although few of the results in the paper can be regarded as being entirely new, the author's broad and sound conception of the subject, and the clarity of the exposition, make the paper a useful contribution.

L. A. MacColl (New York, N.Y.).

Amiantov, I. N.; and Tihonov, V. I. The effect of fluctuation on time range-finder operation. Avtomat. i Telemeh. **19** (1958), 325–333. (Russian. English summary)

The paper deals with problems of measuring range errors caused by fluctuation and with problems of stability of an automatic-tracking system.

Authors' summary.

Raevskiĭ, S. Ya. On dynamical precision of a servosystem including nonlinear block with a polynomial characteristic. Avtomat. i Telemeh. **18** (1957), 1010–1016. (Russian. English summary)

Mean-square error of a servo-system including nonlinear block with a polynomial characteristic is calculated. Disturbances are supposed to be stochastic. The mean-square error is calculated by means of the method of successive approximations using approximate representation of higher moments through the moments of lower orders. *Author's summary.*

★**Caldwell, Samuel H. Switching circuits and logical design.** John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London; 1958. xvii+686 pp. $14.00.

The text of this book is directed to teachers and students of switching circuits, as well as the practicing engineer concerned with logical design problems. The first two chapters deal with applications of switching circuits and the physical nature of relays as components. Boolean algebra as a fundamental mathematical tool is introduced in chapter 3. Here numerous identities are developed for the reduction of the number of components needed in the realization of switching circuits. Chapters 4 to 8 and chapter 11 are concerned with relay combinational switching circuits. Relays are used as a vehicle to prevent obscuring the basic principles while the mathematical theory is being developed. Chapter 9 deals with the use of electron tubes and solid-state devices as specific components in the realization of combinational switching circuits. Chapter 10 treats those aspects of codes that are more useful in designing iterative switching circuits. Chapters 12 and 13 contain the theory and the manipulative rules concerned with the Huffman model [J. Franklin Inst. **257** (1954), 161–190, 275–303; MR **15**, 1009] for the synthesis of sequential circuits. The Mealy model [Bell System Tech. J. **34** (1955), 1045–1079; MR **17**, 436] and the Moore model [Automata studies, Princeton, 1956, pp. 129–153; MR **17**, 1140] of sequential circuits are not considered. Chapter 14 is concerned with the use of electron tubes and solid-state devices in the realization of sequential circuits. Chapter 15 deals with the theory and realization of pulsed sequential circuits.

Each of the chapters 3 to 15 terminate with numerous exercises. These serve both to test the reader's understanding of the material covered, and to treat ideas not discussed, but which follow from the fundamental theory.

A. A. Mullin (Urbana, Ill.).

Adamenko, A. I. Equation of a two-phase asymmetrical asynchronous machine in a stable regime. Dopovidi Akad. Nauk Ukrain. RSR **1957**, 154–160. (Ukrainian. Russian and English summaries)

The author presents equations of an asynchronous two-phase machine in the general case of asymmetry (unequal numbers of coil windings disposed at arbitrary angles). The substitution diagrams are determined for direct and inverse sequence currents, the asynchronous moments being also determined for any harmonic of the stator field. *Author's summary.*

Morosanov, I. S. Optimalizing control methods. Avtomat. i Telemeh. **18** (1957), 1029–1044. (Russian. English summary)

Basic optimalizing control methods are considered. Classification of systems due to the way of searching optimum is presented. The peculiarities of calculating self-oscillations are shown taking relay optimum systems as an example.

Brief recommendations on practical utilization of the said systems are proposed. *Author's summary.*

Kuz'min, L. P. Semigraphical method of determination of relay system characteristics. Avtomat. i Telemeh. **19** (1958), 285–295. (1 insert) (Russian. English summary)

The paper deals with semigraphical method of determination of relay system characteristics $J_1(\omega)$ and $J_y(\omega)$. The said characteristics are required for analysis of periodic regimes in relay systems by means of the frequency method. The determination of the characteristics $J_1(\omega)$ and $J_y(\omega)$ using a certain frequency response of the linear part of the system is described.

Author's summary.

HISTORY, BIOGRAPHY

★**Struik, D. J. Sushe jenshih. [A concise history of mathematics.]** Science Publishing House, Peking, 1956. xvii+174 pp. (33 plates) $0.95.

★**Struik, D. J. Sugaku no rekishi. [A concise history of mathematics.]** Misuzu Shobo, Tokyo, 1958. iv+244 pp. (8 plates) 280 yen.

Translations from the English original reviewed in MR **10**, 173.

Thatcher, A. R. Studies in the history of probability and statistics. VI. A note on the early solutions of the problem of the duration of play. Biometrika **44** (1957), 515–518.

★**Симонов, Н. И. [Simonov, N. I.] Прикладные методы анализа у Эйлера. [Methods of applied analysis in Euler.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1957. 167 pp. 6.15 rubles.

This book deals chiefly with Euler's methods, some of them little known, for integrating ordinary differential equations.

Murata, Tamotsu. French Empiricism. I. Comment. Math. Univ. St. Paul. **6** (1958), 93–114.

The author attempts to describe the evolution of the ideas of Baire, Borel and Lebesgue about the notion of infinity in mathematics. *A. Heyting* (Amsterdam).

Colonnetti, Gustavo. Nel cinquantesimo anniversario di una memoria di Vito Volterra che ha aperta vie nuove alla moderna scienza delle costruzioni. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 95–100.

Anonymous. Obituary: Guido Ascoli. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 5–6. (1 plate)

Tricomi, Francesco G. Guido Ascoli (1887-1957). Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 7–9.

Zin, Giovanni. Ricordo del Prof. Guido Ascoli. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 11–14.

Anonymous. Elenco delle pubblicazioni del Prof. Guido Ascoli. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 15–35.

Anonymous. In memoria di Federigo Enriques. Rend. Mat. e Appl. (5) **16** (1957), 1–2.

Severi, Francesco. Obituary: Federigo Enriques. Rend. Mat. e Appl. (5) **16** (1957), 3–8.

Sansone, Giovanni. Obituary: Federigo Enriques. Rend. Mat. e Appl. (5) **16** (1957), 9–11.

Togliatti, Eugenio Guiseppe. Obituary: Federigo Enriques. Rend. Mat. e Appl. (5) **16** (1957), 12–22.

Severi, Francesco. Commemorazione del Prof. Luigi Fantappiè all'Istituto di Alta Matematica. Rend. Mat. e Appl. (5) **16** (1957), 140–142.

Fichera, Gaetano. La vita matematica di Luigi Fantappiè. Rend. Mat. e Appl. (5) **16** (1957), 143–160.

Ghizzetti, Aldo. Obituary: Ugo Amaldi. Rend. Mat. e Appl. (5) **16** (1957), 511–514.

Anonymous. Aurel Wintner, 1903-1958. Amer. J. Math. **80** (1958), 1.

★ **Segre, Corrado. Opere.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Vol. 2. Edizioni Cremonese, Rome, 1958. xviii+438 pp. (1 plate) 4000 Lire.

This second volume (the first was reviewed in MR **18**, 982) contains 19 articles and a photograph. These articles, written from 1886 to 1924, may be classified, as is stated in the preface, under the titles of "geometria differenziale" and "enti immaginari".

★ **Bianchi, Luigi. Opere.** A cura dell'Unione Matematica Italiana e col contributo del Consiglio Nazionale delle Ricerche. Vol. 9: Geometria degli spazi di Riemann. Vol. 10: Ricerche varie. Edizioni Cremonese, Rome, 1958. Vol. 9: 264 pp.; Vol. 10: 79 pp.; bound as one volume. 3500 Lire.

Vol. 9 contains 9 articles, written from 1898 to 1917, with an introduction by A. Maxia. Vol. 10 contains 8 articles, written from 1901 to 1927, with an introduction by P. Tortorici.

MISCELLANEOUS

★ **Bakst, Aaron. Mathematical puzzles and pastimes.** D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York. vii+206 pp. $4.00.

Fifteen chapters of entertaining problems solvable by elementary means. There are many diagrams and solutions are given.

★ **Goodell, John D. The world of ki.** Riverside Research Press, St. Paul, Minn., 1958. iv+215 pp. Paperbound $3.00; clothbound $4.50.

A textbook on the game of Go with interspersed sections (in non-mathematical language) on the general theory of games, and in particular on the recent mathematical development of this theory.

★ **Ohara, E. Japanese chess: the game of Shogi.** Charles E. Tuttle Co., Rutland, Vt.-Tokyo, 1958. xvii+182 pp. $2.75.

Introduction to the rules and strategy of Shogi. This game, popular in Japan and little-known elsewhere, is distinguished from other versions of chess in that a piece which one has captured may be subsequently replaced on the board as one's own.

★ **Holtmann, Friedrich. Mathematik. Bd. 1: Arithmetik.** 4te, verbesserte Aufl. Fachbuchverlag, Leipzig, 1958. xiv+322 pp. DM 6.00.

A treatment of elementary analysis, through differential and integral calculus, for technical workers.

★ **Sears, Francis Weston. Mechanics, wave motion, and heat.** Principles of Physics Series. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1958. xiii+664 pp. $9.50.

This clearly written text gives numerous examples and problems, for the application of the calculus and of vector analysis to such subjects as harmonic motion, hydrodynamics, and thermodynamics.

★ **Bronstein, I. N.; und Semendjajew, K. A. Taschenbuch der Mathematik: für Ingenieure und Studenten der technischen Hochschulen.** B. G. Teubner Verlagsgesellschaft, Leipzig, 1958. xii+548 pp. (1 insert) DM 22.50.

A translation by V. Ziegler of the 6th edition of the Russian original (Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956]. The section headings are: Tabellen und Kurven, Elementare Mathematik (Naherungsrechnung, Algebra, Geometrie, Trigonometrie), Geometrie (Analytische, Differential), Analysis (Einführung, Differentialrechnung, Integralrechnung, Differentialgleichungen), Ergänzende Kapitel der Analysis (Funktionen einer komplexen Veränderlichen, Vektorrechnung, Fourierreihen, Variationsrechnung), Auswertung von Beobachtungsergebnissen (Grundzüge der Wahrscheinlichkeitsrechnung und der Theorie der Beobachtungsfehler, Empirische Formeln und Interpolation). The chapter on the calculus of variations was written especially for the German edition.

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